Section 4.1 – Inverse Functions

**Inverse Relations:**
- When finding an **inverse relation** you are looking to 'undo' the process that was done.
- For example, the inverse of the square function \( f(x) = x^2 \) is the square root function \( g(x) = \sqrt{x} \) because \( f(2) = 4 \), and \( g(4) = 2 \).

Notice that in the previous example you are taking the ordered pair \((2,4)\) for \( f(x) \) and the ordered pair \((4,2)\) for \( g(x) \). In fact, interchanging the first and second coordinates of each ordered pair in a relation is another way to define an **inverse relation**.

To find an inverse relation for \( y = \text{relation of } x \), interchange the \( x \)'s and \( y \)'s in the equation and (if possible) solve for \( y \).

**Example.** Find the equation of the inverse of \( y = x^2 - 6x + 9 \)

\[
\begin{align*}
x & = y^2 - 6y + 9 \\
x & = (y - 3)^2 \\
y & = \pm \sqrt{x} + 3
\end{align*}
\]

Is the original problem a function? Does it pass the vertical line test? Is the inverse a function?

**Inverse Functions and One-to-One:**
- Everything we looked at above holds for functions as well. You can find an inverse function by switching the order of all ordered pairs, or you can switch the \( x \)'s and \( y \)'s in the equation.
- The notation for inverse functions is a little bit different, because for a function \( f(x) \), its inverse is written as \( f^{-1}(x) \). Note, this does NOT mean \( 1/f(x) \), but \( f \) inverse!
Notice in the problem given above, \( f(x) = x^2 - 6x + 9 \) is a function. But the inverse, \( \pm \sqrt{x + 3} \) is NOT a function. How did this happen?

Recall that for \( y(x) \) to be a function, each \( x \) has only one \( y \)

A \( y(x) \) is **one-to-one** if it is a function, and also satisfies that each \( y \) maps back to only one \( x \)

For one-to-one there is a correspondence between each \( x \) and \( y \), and it has to pass not only the vertical line test, but also a horizontal line test

*Example, are each of the following graphs an example of a function? Is it one to one?*

(a) is not a function, so it also cannot be 1-1
(b) is a function but is not 1-1
(c) is a function and is also 1-1

The following functions are always one-to-one:
Linear, square roots

The following functions are not ever one-to-one:
Quadratic, absolute value
Example. Graph \( f(x) = \frac{5x-3}{2x+1} \), determine if it is 1-1, if so find the inverse

What is the root of \( f(x) \)?
When \( 5x - 3 = 0 \), or \( x = 3/5 \)

What is the undefined value for \( x \)?
When \( 2x + 1 = 0 \), or \( x = -1/2 \)

It passes the vertical and horizontal line tests
It is one to one

Find the inverse:
\[
\begin{align*}
x &= \frac{5y-3}{2y+1} \\
2xy + x &= 5y - 3 \\
y(2x - 5) &= -x - 3 \\
y &= \frac{-x-3}{2x-5}
\end{align*}
\]

NOTE: The domain for \( f(x) \) is all values except \(-1/2\). The range is all values except \(5/2\)
The domain for \( f^{-1}(x) \) is all values except \(5/2\), and the range is all values except \(-1/2\)

What is the general relationship between the domain and ranges for functions and their inverses?

- Since an inverse function 'undoes' the original function, if we compose the two [i.e. take \( f \circ f^{-1}(x) \) or \( f^{-1} \circ f(x) \)] we should get \( x \)

- Example. Show \( f(x) = \frac{x+5}{4} \), \( f^{-1}(x) = 4x - 5 \) are inverses
\[
f[f^{-1}(x)] = \frac{[4x-5]+5}{4} = \frac{4x}{4} = x
\]