Section 3.3 – Polynomial Division. Remainder and Factor Theorems

• What is the process used in long division?
i.e. \( \frac{650}{5} \)

\[
\begin{array}{c c c c c c}
5 & 650 \\
130 & 650 \\
5 & 15 \\
15 & 0
\end{array}
\]

• What happens if the value results in a remainder?
i.e. \( \frac{235}{5} \)

\[
\begin{array}{c c c c c c}
46 & 233 \\
20 & 33 \\
30 & 3
\end{array}
\]

\[
5 \div 233 = 46 + \frac{3}{5}
\]

• Dividing polynomials follows the same procedure
First write out the polynomial in decreasing order with all terms (even if they have a 0 coefficient)

\[
\begin{array}{c c c c c c}
x-1 & x+1 & x^2 + 0x-1 \\
x+1 & x^2 + x \\
-1 & 0
\end{array}
\]

\( x \cdot (x + 1) \) put the result here and subtract the lines

\[
\begin{array}{c c c c c c}
-1 & x-1 \\
-1 & 0
\end{array}
\]

Check: \((x-1)(x+1) = x^2 - 1\)
This process can be simplified by what is known as **synthetic division**. Let's follow the steps with an example.

For the function \( f(x) = x^4 + 7x^3 + 8x^2 - 28x - 48 \) show \((x - 2)\) is a factor.

We start with the root, which for the factor \((x - 2)\) is +2.

1. Put 2 (the root) in the corner
2. Make a table
   - all coefficients in decreasing order (even 0s)
3. Bring down the 1 (the first coefficient)
4. Repeat the following until you get to the end
   - Multiply the number below with the root
   - Place it under the next coefficient
   - Add, put the result below
   - Continue

Since the number at the end is a zero, you have shown that \( x - 2 \) is a factor of \( f(x) \)

In other words \( f(x) = (x^3 + 9x^2 + 26x + 24)(x - 2) \) or \( \frac{f(x)}{(x - 2)} = x^3 + 9x^2 + 26x + 24 \)

**Example, for \( f(x) \) given above, show that \((x - 1)\) is NOT a factor**

\[
\begin{array}{cccccc}
1 | & 1 & 7 & 8 & -28 & -48 \\
 & & 1 & 8 & 16 & -12 \\
\end{array}
\]

\[1 \ 8 \ 16 \ -12 \ -60\]

Because the remainder is not zero, 1 is not a root and \((x - 1)\) is not a factor.

Therefore, we have that \( \frac{f(x)}{(x - 1)} = (x^3 + 8x^2 + 16x - 12) + \frac{-60}{x - 1} \)

We can write this in another form: \( f(x) = (x^3 + 8x^2 + 16x - 12)(x - 1) - 60 \)

**Example. Determine if \((x - 1)\) is a factor of \( P(x) = x^4 + 6x^3 \)**

\[
\begin{array}{cccc}
1 | & 1 & 6 & 0 & 0 \\
 & & 1 & 7 & 7 \\
\end{array}
\]

\[1 \ 7 \ 7 \ 7\]

No, it is not a factor because the remainder is not zero.

\( P(x) = (x^3 + 7x^2 + 7x + 7)(x - 1) + 7 \)
• Example. Determine if $-4$ and $2$ are roots of $f(x) = 3x^3 + 11x^2 - 2x + 8$

There is an excel sheet online that will help you check your work…

For the value $-4$:

<table>
<thead>
<tr>
<th>$x^6$</th>
<th>$x^5$</th>
<th>$x^4$</th>
<th>$x^3$</th>
<th>$x^2$</th>
<th>$x^1$</th>
<th>$x^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>11</td>
<td>-2</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-12</td>
<td>4</td>
<td>-8</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>-1</td>
<td>4</td>
</tr>
</tbody>
</table>

The given value ___ is a root of the polynomial

The factorization would be

$$f(x) = (x - (-4)) (3x^2 - x + 2) (x + 4)$$

For the value $2$:

<table>
<thead>
<tr>
<th>$x^6$</th>
<th>$x^5$</th>
<th>$x^4$</th>
<th>$x^3$</th>
<th>$x^2$</th>
<th>$x^1$</th>
<th>$x^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>11</td>
<td>-2</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>34</td>
<td>64</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>17</td>
<td>32</td>
</tr>
</tbody>
</table>

The given value ___ is not a root of the polynomial

The factorization would be

$$f(x) = (x - 2) (3x^2 + 17x + 32) (x - 2) + 72$$