Section 3.1 – Polynomial Functions and Models

Classifying Polynomials:

- Recall: a polynomial in one variable is an expression of the form
  \[ a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 , \]
  where the \( a_i \)'s are real number coefficients. For nonzero \( a_n \), the expression is said to be of \( n \)th degree (the highest power is \( n \)), the leading term is \( a_n x^n \) and the leading coefficient is \( a_n \).

- Examples of polynomials that are common

<table>
<thead>
<tr>
<th>Degree</th>
<th>Name</th>
<th>Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Constant</td>
<td>( f(x) = c )</td>
</tr>
<tr>
<td>1</td>
<td>Linear</td>
<td>( f(x) = mx + b )</td>
</tr>
<tr>
<td>2</td>
<td>Quadratic</td>
<td>( f(x) = ax^2 + bx + c )</td>
</tr>
<tr>
<td>3</td>
<td>Cubic</td>
<td>( f(x) = ax^3 + bx^2 + cx + d )</td>
</tr>
</tbody>
</table>

- Example. Classify \( f(x) = 15x^2 - 10 + 0.11x^4 - 7x^3 \)
  The highest power is 4, so it is a 4th degree polynomial. A quartic
  The leading term is 0.11\( x^4 \).

Leading Behavior:

- It is often useful to talk about a function as the independent variable (usually \( x \)) gets really large (\( x \to \infty \)) or really small (\( x \to -\infty \)).
- As the independent variable gets really large in magnitude (\( x \to \pm \infty \)), the leading term dominates the equation. All polynomials (except the constant function) will tend to \( \pm \infty \) as \( x \) gets large in magnitude. Why?
- To determine if it goes to \( +\infty \) or \( -\infty \), look at the behavior of the leading term, and evaluate it with a positive and a negative.

- Example, what happens to \( f(x) = -x^4 + 3x^2 + 3x \) when \( x \to \pm \infty \)?
  The leading term is \( -x^4 \)
  As \( x \to +\infty \), \( f \to -(+\infty)^4 = -\infty \)
  As \( x \to -\infty \), \( f \to -(\infty)^4 = -\infty \)

- Example, what happens to \( f(x) = 12x^5 + 3x^4 + 3x \) when \( x \to \pm \infty \)?
  The leading term is \( 12x^5 \)
  As \( x \to +\infty \), \( f \to 12(+\infty)^5 = +\infty \)
  As \( x \to -\infty \), \( f \to 12(-\infty)^5 = -\infty \)

- Polynomials can be classified according to their degree (highest power) as even or odd (note that this may not have same meaning as even and odd regarding symmetry).
  – If a polynomial has an even degree, it goes in the same direction for \( x \to \pm \infty \)
  – If a polynomial has an odd degree, it goes in opposite directions for \( x \to \pm \infty \)

You can use the same test as above to confirm this for yourself.
• **Example.** Describe the end behavior for \( f(x) = \frac{1}{4}x^4 + \frac{1}{2}x^3 - 6x^2 + x - 5 \)

The highest power is 4, so it is a 4th degree polynomial.
It has even degree (same direction at each end)
The leading term is \( \frac{1}{4}x^4 \)

As \( x \to +\infty \), \( f \to \frac{1}{4}(+\infty)^4 = +\infty \)

As \( x \to -\infty \), \( f \to \frac{1}{4}(-\infty)^4 = +\infty \)

**Roots of Polynomials:**

• When you set a polynomial \( f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \) equal to zero, it is equivalent to finding the **roots** (or **zeros**) of the function. In other words, what values of \( x \) make \( f(x) = 0 \). This is where it would cross the \( x \) axis

• For an \( n \)th degree polynomial, you will have \( n \) roots. Keep in mind these roots do not have to be unique, and they do not have to be real numbers. This means you will cross the \( x \) axis **at most** \( n \) times

• **Example, find the roots of** \( f(x) = x^3 + 10x^2 + 25x \)

\[ f(x) = x(x^2 + 10x + 25) = x(x + 5)^2 \]

The roots of \( f(x) \) are 0 and –5. Notice –5 is a repeated root

**Critical Points of Polynomials:**

• Polynomials of \( n \)th degree can change direction (from increasing to decreasing, and vice versa) at most \( n - 1 \) times

• For example, a polynomial of degree 1 (a line) doesn't change direction at all. A polynomial of degree 2 (a quadratic) changes direction once

• Where a polynomial changes directions is not dependent on where it crosses the \( x \) axis! It has to do with what are known as **critical points** on the graph. (This can be studied in any elementary calculus I class)

• The book presents a method for graphing that is less than desirable. In short, it says to:
  – use the leading term test
  – find the roots
  – test the areas divided by the roots
  – find \( f(0) \)
  – sketch
That method fails miserably with the function $f(x) = 2x^3 - 9x^2 + 12x$
Leading term says as $x \to +\infty$, $f \to \infty$ and as $x \to -\infty$, $f \to -\infty$

Now $f(x)$ can be factored as $f(x) = x(2x^2 - 9x + 12)$. Why can't it be factored further?
So the only root of the equation is $x = 0$
To the left of 0 $f$ is positive, to the right of 0 $f$ is negative
$f(0) = 0$
If we didn't worry about anything else, all this information would give us is a picture similar to:

But in reality the graph looks like:

What this method is missing is an analysis of the critical points of the graph. There will be at most $n - 1$ critical points in an $n$th degree polynomial. You will not be required to find them, but you should know how many there could be, and how they effect the shape of the graph.

Example, what do the critical points tell you about the previous example?
Since it is a 3rd degree polynomial, there are at most 2 critical points.
This means that the graph could change direction at least twice
(from the picture you can see it goes from increasing to decreasing and back to increasing again)
• **Example.** Find zeros and multiplicity for \( f(x) = \left(x + \frac{1}{2}\right)(x + 7)(x + 7)(x + 5) \)
  \( f(x) \) has roots at: \(-1/2\) (mult 1), \(-7\) (mult 2) and \(-5\) (mult 1)

• **Example.** Find zeros and multiplicity for \( f(x) = 3x^3 + x^2 - 48x - 16 \)
  \( f(x) = x^2(3x + 1) - 16(3x + 1) = (x + 4)(x - 4)(3x + 1) \)
  Roots at \(4\) (mult 1), \(-4\) (mult 1) and \(-1/3\) (mult 1)

• **Example.** Find roots, behavior and number of cps for \( h(x) = x^5 - 5x^3 + 4x \)
  As \( x \to \infty, h \to \infty \) and as \( x \to -\infty, h \to -\infty \)
  There are at most 4 critical points
  There are at most 5 roots. \( x^5 - 5x^3 + 4x = x(x^4 - 5x^2 + 4) = x(x + 2)(x - 2)(x^2 + 4) = 0 \)
  They are: 0, -2, and 2

**The Intermediate Value Theorem:**
- The intermediate value theorem is used to 'trap' roots between two values. It is useful in estimation, when you are not quite sure where the root is but want to estimate it
- For any polynomial \( P(x) \), there is at least one root \( c \) in between the values \( a \) and \( b \) if \( P(a) \) and \( P(b) \) have opposite signs

\[
\begin{align*}
P(a) &> 0 \\
P(c) &= 0 \\
P(b) &< 0
\end{align*}
\]

• **Example.** Show if \( f(x) = x^3 + 3x^2 - 9x - 13 \) has a zero between \( a = 1 \) and \( b = 2 \)
  \( f(x) \) is a polynomial
  \( f(a) = f(1) = 1^3 + 3 \cdot 1^2 - 9 \cdot 1 - 13 = -18 \)
  \( f(b) = f(2) = 2^3 + 3 \cdot 2^2 - 9 \cdot 2 - 13 = -11 \)
  No, there may not be a root between \( a \) and \( b \) because \( f(a) \) and \( f(b) \) are the same sign
  There is one, by the way, between 2 and 3 because \( f(3) = 3^3 + 3 \cdot 3^2 - 9 \cdot 3 - 13 = 14 \)