Tableau for Incompatibility Semantics:
   Soundness and Completeness

Let's return to incompatibility semantics, with an eye toward determining how well its intricacies are illustrated by two-sided tree proofs characterized “pragmatically.”

Soundness:

1. The soundness of the method with respect to incompatibility semantics is rather easily demonstrated. For if there is an interpretation or incompatibility frame that permits the simultaneous affirmation of all the premises and also the denial of a conclusion, one can straightforwardly show that as we apply the rules of development to a tree that begins with those premises on the left and that conclusion on the right, that same incompatibility frame will permit the simultaneous affirmation of all the formulas along the left of some path of that tree alongside the denial of all the formulas on the right of that very same path. Since a path closes only when the same formula appears both on the left and the right, and since one may not affirm and deny one and the same formula, we know then that such a path will never close, and so we’ll never have a tree proof of that conclusion from those premises.

2. The actual proof is altogether similar to demonstrations of soundness with respect to truth-functional understandings of the classical sentential operators. We proceed to show how applying the rules of development to consistent paths will inevitably yield at least one path that is also consistent:

   (&L): Since a conjunction incompatibility entails each of its conjuncts, anything that would stand in the way of placing a conjunct along a consistent path containing a conjunction on the left would have already disrupted the consistency of having that conjunction on the left. So we may extend the path by placing both conjuncts along the left (in accord with the tableau rule for conjunction on the left) without loss of consistency.

   (&R): Suppose one begins with a consistent path with a conjunction on the right. By definition of the conjunction, any grounds for denying that conjunction would also be grounds for denying both of the conjuncts together. But that means that one must always remain in position to deny at least one of the conjuncts. And so extending our path by placing one or the other conjunct on the right (in accordance with the rule for conjunction on the right) is still bound to leave at least one coherent path.

   (VL): Since the incompatibility set of a disjunction is the intersection of the incompatibility sets of both of its disjuncts, we know that nothing on that path would stand in the way of both of the disjuncts, and so we can rest assured that we may add one or the other to the left of that path without loss of consistency.

   (VR): Since a disjunction is incompatibility entailed by each of its disjuncts individually, any grounds for coherently denying the disjunction would also have to be grounds for denying each disjunct. So we may extend a coherent path in accord with the rules for disjunction on the right without loss of consistency.

   (~L): Now suppose that a negation is on the left of a coherent path. Thus it must be possible to affirm everything on the left while denying everything on the right. Recall that ~φ is the minimal incompatible of φ, a sentence whose incompatibility set (or incompatibility content) is to be understood as the intersection of the incompatibility sets of everything incompatible with φ. Since φ must be in all of those incompatibility sets, it follows that φ at least must belong in the incompatibility set of ~φ. Thus a
coherent commitment to \( \neg \phi \) already precludes a commitment to \( \phi \). So developing the path by putting \( \phi \) on the right will not change the fact that we have a path in which one may coherently be committed to everything on the left and precluded from everything on the right. Hence developing a path according to the rule for negation on the left preserves coherence.

\((\neg \text{R})\): Finally, let us suppose that some negation \( \neg \phi \) is on the right side of a consistent path. By definition anything incompatible with this negation must already incompatibility entail the negated formula \( \phi \). That means that anything incompatible with \( \phi \) would also have to be incompatible with any grounds for denying \( \neg \phi \). Since we are assuming these formulas to be on a consistent path and so may deny \( \neg \phi \), we may not be affirming anything incompatible to \( \phi \). And so placing it on the left of our assumed consistent path in accord with the tableau rule for negation on the right will not alter the path’s consistency.

Completeness:

3. Recall that the key to our proof of tableau completeness rests in the fact that one can so readily read off a counterexample to a sequent or argument form from a non-closing path of a “failed” attempt to give it a proof. Pleasantly, this feature of trees carries over to the incompatibility understanding of the sentential connectives. This might at first seem surprising since the construction of an incompatibility frame is a comparatively complicated exercise, which requires us to pay attention to material, inferential relationships between atomic sentences rather than simply assign them independent truth values. However, we can also understand trees as displaying relevant material incompatibility relationships between sentential (and perhaps subsentential) contents, and so one can look to the atomic sentences along a non-closing path of a “failed” proof tree to generate the minimal constraints of an incompatibility frame for it to as a relevant counterexample.

4. Here we simply require that the set of every atomic formula on the left of the chosen path falls outside the set of incompatible formulas specified in an incompatibility frame, while we stipulate that that set is incompatible with each and every atomic formula that appears on the right. [Such incompatibility might be a result of taking that atomic formula on the right to be pairwise incompatible with any single atomic on the left. The persistence constraint on incompatibility frames will take care of the rest.] One can then show on an incompatibility frame obeying these minimal constraints, that by affirming all of the atomics on the left (which by stipulation is a coherent set of commitments), one may then affirm all of the formulas on the left, and also be precluded from affirming any of the formulas on the right. Observe that it will be enough here to argue of all the formulas on the left that they are incompatibility entailed by the set of atomics on the left. For if something is incompatibility entailed by that set, then adding it to one’s stock of commitments will not generate any new incompatibilities or preclude anything that wasn’t already precluded before by one’s prior commitment to those atomics.

5. We now proceed to our induction, which will be on the length of formulas appearing on either side of our non-closing path. Our inductive hypothesis will be that for every formula on the path shorter than an arbitrary one, \( \chi \), the set of all of atomic formulas on the left under this interpretation will incompatibility entail that formula if it is on the left and be inconsistent with that formula if it is on the right.

(i) Base Case; \( \chi \) is atomic (and the inductive hypothesis is moot):
We are starting with commitment to all of the atomics on the left, which is stipulated under this interpretation/incompatibility frame to be a compatible set of commitments. Trivially, since this set includes all of the atomics on the left, it will incompatibility entail each of them individually. Furthermore, our incompatibility frame stipulates that that set is inconsistent with each of the atomics on the right.

(ii) $\chi$ is a negation: $\neg \phi$

Suppose we have a negation on the left of our non-closing path. By our rules of tree development, the formula $\phi$ that it negates must be on the right. That formula is clearly shorter than $\chi$, and so by our inductive hypothesis, the set of all the atomics on the left is incompatible with it. But by the incompatibility definition of negation, anything that is incompatible with that formula $\phi$ must also incompatibility entail its negation $\neg \phi$. And so the set of atomics on the left must incompatibility entail that negation.

Finally, suppose we have a negation on the right of the path. By our rules of tree development, the formula $\phi$ it negates must be on the left, and so by the inductive hypothesis, that formula $\phi$ is incompatibility entailed by the set of all the atomics on the left. Since $\phi$ is clearly incompatible with its negation and anything incompatible to $\phi$ would be incompatible to the joint set of atomics on the left, then this joint set must also preclude the negation.

(iii) $\chi$ is a conjunction: $(\phi \& \psi)$

First suppose that $\chi$ is on the left of the path. By our rules of tree development, both $\phi$ and $\psi$ must also lie on the left. But these formulas are clearly shorter than $\chi$, so our inductive hypothesis holds of them both. Thus the set of all of the atomics on the left under this interpretation incompatibility entails each individually. Now our definitions of incompatibility entailment and conjunction obey the basic rule for conjunction: $\Gamma \models (\phi \& \psi)$ just in case $\Gamma \models \phi$ and $\Gamma \models \psi$. Applying this rule from right to left (the proof of which I suppress here, because it requires two applications of the “cut”), we can conclude that since the set of atomics on the left incompatibility entails each of the conjuncts, that set must also incompatibility entail the conjunction as well.

Now suppose that $\chi$ is on the right side of the path. By our rules of development, our open path must be one in which either $\phi$ or $\psi$ is on the right of the path. But both of these formulas are clearly shorter than $\chi$, so our inductive hypothesis holds, and thus one or the other must be precluded by commitment to all of the atomics on the left under our interpretation. But the incompatibility sets of either of these conjuncts are by definition subsets of everything incompatible with their conjunction. And so if a conjunct is precluded, so too must the conjunction also be precluded.

So commitment to all of the atomics on the left under this interpretation entails commitment to $\chi$ if it is on the left and preclusion to $\chi$ if it is on the right, which is exactly what we want to show.

(iv) $\chi$ is a disjunction: $(\phi \vee \psi)$

Suppose we have a disjunction on the left of the path. By the rules of tree development, at least one of its disjuncts must also be on the left of that path. But the inductive hypothesis would apply to that disjunct, meaning that on our incompatibility frame, that disjunct is incompatibility entailed by joint affirmation of all the atomics on the left. However, since disjuncts already incompatibility entail their
disjunctions, then that means that the disjunction would also have to be incompatibility entailed by joint affirmation of all these atomics on the left.

Suppose instead that we have a disjunction on the right of the path. By the rules of tree development, both of the disjuncts must also be on the right, and the inductive hypothesis would hold of both. But that means that joint affirmation of all the atomics on the left would force us to deny both of those disjuncts, meaning in turn that that set of atomics is in the incompatibility set of both disjuncts. But by the definition of incompatibility disjunction, that set must also be incompatible with entire disjunction. And so affirmation of the atomics on the left would force one to deny that disjunction as well.

6. Thus completes our demonstration that the Tableau is complete with respect to semantics centered around the notion of incompatibility. Together with soundness, the Tableau system of proof is fully adequate with respect to incompatibility semantics.