Introducing the Double Turnstile

1. The lifeblood of logic is \textit{entailment}. This is the familiar phenomenon of a certain statement or sentences \textit{following from} (or being a \textit{consequence of}) other statements. Their affirmation commands or obliges one to accept others.

For example, the statement “Las Vegas is to the west of Pittsburgh.” follows from “Pittsburgh is to the east of Las Vegas.” One may not affirm the latter without committing oneself to accepting the latter. [And, it so happens, vice versa.] This entailment is secured by virtue of what it means for one location to be ‘east’ (or ‘west’) of another.

Similarly (though not equivalently) the statement “Aristotle is a logician.” follows jointly from “Aristotle is a student of Plato.”, “Any student of Plato is a Philosopher.”, and “All Philosophers are logicians.” Here, the entailment seems secured by the meanings of ‘is’ and ‘all’, as well as by \textit{the form} (or order) that these sentences take.

Those who study this phenomenon (that is, logicians!) often use the double turnstile (|=) to symbolize or to stand for this phenomena. In a sense, then, we can say that the primary (if not sole) concern of logic resides with the double turnstile.

2. Syntax of |=:

The double turnstile participates in expressions that we in the logic trade call \textit{sequents}. The double turnstile is an \textit{infix} that is placed between expressions standing for either single statements or sentences and sets of sentences or statements. By convention, we symbolize sets of statements with capital Greek letters and individual statements with lower-case Greek letters.

Thus sequent expressions will typically take the following forms: $\Gamma |\!\!\!\!=\alpha$, $\Phi |\!\!\!\!=\omega$, $\Pi |\!\!\!\!=\nu$, or $\Delta,\phi |\!\!\!\!=\psi$.

3. The semantics (or “meaning”) of |=:

Sequent expressions assert that an \textit{entailment} relation holds between the set of sentences on the left and right-hand sides of the double turnstile. How, then, should we unpack this notion of entailment?

Here is an initial stab at understanding what is said by a sequent expression:

(Def $\models$) \textit{It is not possible for everything on the left of the double turnstile to be true while something on the right is false.}

4. The Traditional Understanding of |=:

While sets of sentences can belong on the right-hand side of sequents, we will mostly be interested in sequents of the form $\Gamma |\!\!\!\!=\alpha$, in which the left-hand side contains an expression standing for a set of sentences, and the right-hand side contains an expression for a \textit{single} statement. For such a sequent can be understood as expressing something familiar about an argument which has all the statements on the left as premises, and the single statement on the right as conclusion.
When applied to a sequent of the form $\Gamma \models \alpha$, our understanding of the double turnstile yields the following:

(Simplified Def $\models$) It is not possible for all the sentences in $\Gamma$ to be true while the sentence $\alpha$ is false.

And this, I imagine, would line up with the understanding of (logical) validity one might have gotten in an earlier logic course.

5. The Traditional Understanding Again

Among other things, however, this unpacking relies on our having an understanding of which sentences may be true or false together. This requisite understanding of what is or is not possible is usually supplied by the logician’s notion of an interpretation. So here we get another, more refined understanding of sequent expressions such as $\Gamma \models \alpha$.

(Refined Def $\models$) There is no interpretation (of the sentences in $\Gamma$ and $\alpha$) in which every sentence of $\Gamma$ is true and the sentence $\alpha$ is false.

For the bulk of this course, we will deploy this understanding of sequent expressions. COMMIT IT TO MEMORY!!

6. An Alternate Understanding

Observe that this specific understanding of sequent expressions (and thus, of logic itself!) gives pride of place to the notion of a statement or sentence’s being true. However, some of us in the business happen to be a bit leery of a metaphysical notion like truth playing any great explanatory role or doing any philosophical heavy-lifting. As a result, we are compelled to offer an alternate understanding of what is expressed by sequents. Fortunately for us, this can be done in terms of the “pragmatic” acts or attitudes of affirming and denying statements or the “deontic” states of being committed to statements and being precluded from (being committed to) them. So here is an alternative unpacking of the double turnstile that doesn’t lean on the notion of truth.

(Alternative Def $\models$) One may not affirm (or be committed to) every statement on the left while also denying (or being precluded from) some statement on the right.

Notice that on this alternate characterization, the roles played earlier by truth and falsity are instead played by the notions of affirmation and denial. On occasion, I will speak about this alternative as the “pragmatic,” “deontic,” or “incompatibility” approach to logic. However, please keep in mind that this is not the understanding that I’ll be working with for most of this course. Though I happen to endorse this route of understanding the fundamental notions of logic, I recognize that such an approach would be idiosyncratic. I only offer it as a reminder that philosophical logic can be elaborated and pursued (pretty much as it always has been) in a manner that doesn’t speak much about truth or falsity at all.

7. There are some “degenerate” forms of sequent expressions, in which either the left or the right side is left empty. Think of these as sequents flanked on one side or the other by the empty set. They are to be understood in accordance with our general understanding of sequent expressions, but without paying any consideration to the empty side. Thus, the sequent $\models \psi$ simply expresses the idea that there is no interpretation in which the sentence $\psi$
is false. In other words, it corresponds to the idea that $\psi$ is a \textit{tautology}, which should be familiar to you from your introductory logic course.

Likewise, the sequent $\Gamma \vdash \_|$ expresses the idea that there is no interpretation in which everything in $\Gamma$ is true. That is, this is a convenient way of saying that the set of sentences represented by $\Gamma$ is \textit{inconsistent}.

[For what it’s worth, the completely degenerate sequent consisting of the double turnstile by itself (or flanked by the empty set on both sides) may be understood simply as saying “There is no interpretation.” Since logic is in the very business of interpreting languages, we can understand that sequent as asserting something that we logicians will regard as incorrect.]

8. In this way, we can extend the language of sequents so that it can say logically interesting things, not just of complete arguments (comprising both sets of premises and conclusions), but also of individual sentences or of sets of sentences. And indeed if we also allow ourselves the ability to negate or to deny sequent expressions (which we can symbolize simply by drawing a line through the double turnstile, much as we symbolize inequality by striking through an equals sign), we can then deny that certain entailments hold, and so also express the idea that certain sentences are \textit{contingent} or \textit{contradictory}, or that certain sets of sentences are \textit{consistent}.

9. A few exercises:

1. Using the language of sequents and denials thereof, express the idea that a certain \textit{set of sentences} is \textit{consistent}.

2. Using that same language, express the idea that a certain \textit{single sentence} is \textit{contradictory}. [Be sure not to appeal to the notion of that sentence’s negation!]

3. Using that same language, express the idea that a certain sentence is \textit{contingent}. [Hint: this will require two separate sequent expressions.]

4. Given what was said in section 8 above, what is meant by $\vdash \_?$ Should we regard it as correct or incorrect?