An alternative understanding of interpretations: Incompatibility Semantics

1. In traditional (truth-theoretic) semantics, interpretations serve to specify when statements are true and when they are false. For propositional logic, interpretations consist of individual and independent assignments of truth values to atomic propositions, from which one may determine the truth values of compound propositions. Thus the question of semantic entailment (the double turnstile) concerns whether certain statements can be true or false together. However, this approach does not offer us a particularly rich or nuanced understanding of the semantic values of atomic propositions. Distinct propositions will have precisely the same semantic value as long as they are assigned the same truth-value, despite the fact that, intuitively, their meaning or content is quite distinct. The trouble is that truth values seem to be an especially crude way of understanding a proposition’s content. About the only thing one can do with them is to “plug” them into truth functions.

2. However, I pointed out (way back when) that we might instead understand entailment in terms of the pragmatic speech acts of assertion and denial (or the deontic statuses of commitment and preclusion) or, more broadly, affirmation and rejection, or acceptance and refusal. A set of statements can entail another just when assertion, affirmation, or acceptance of the former precludes the denial, rejection, or refusal of the latter.

3. As a digression, such a pragmatic account of entailment might well offer us the resources to begin to understand evidently valid reasoning involving statements such as imperatives or commands, which we don’t typically regard as being true or false (such statements are sometimes said to have “expressive” rather than “descriptive” content). For instance, even though commands don’t have truth values, they nevertheless participate in some sort of logical reasoning governed by entailment relations. Consider the following straightforward case of “imperative disjunctive syllogism.” Suppose Carol acknowledges or accepts the two following commands:

   Alice: Hey Carol, buy me a moon pie or a goo-goo cluster!
   Bob: Don’t buy her a moon pie!

If Carol accepts these commands (something she can indicate by simply nodding or saying OK), she can rightly infer that she should not refuse to buy Amy a goo-goo cluster. A similar example can be formulated deploying of form of “imperative Modus Ponens”:

   Alice: Carol, go to the store!
   Bob: Oh hey, Carol, if you go to the store, buy me some Zagnuts!
   Carol: OK you two!

Once again, Carol’s acceptance of these two commands (the second of which is conditional) logically precludes her from refusing to buy Bob some Zagnuts. In short, the lesson here is that we might well like to extend our account of logical entailment to capture evident entailment relationships that hold not just between statements that are “merely truth-bearing” (or those that one might affirm or deny), but also to kinds of statements that can, more generally, simply be accepted or refused.
4. Incompatibility semantics is a first step in getting us away from “the tyranny of truth.” In incompatibility semantics, an interpretation consists in the specification of an incompatibility frame, which serves to tell us directly which sentences may be affirmed or denied together. At root, an incompatibility frame is a structure that lists which sets of sentences are to be understood as internally incompatible (subject, usually, to a constraint called “persistence”). The idea is that if a set appears in the incompatibility frame, then one is precluded from affirming (or being jointly committed to) all the members of that set. Unlike traditional interpretations of propositional logic, where atomic propositions are assigned truth values independently of one another, incompatibility frames can (and do!) relate atomic propositions to one another in material incompatibilities; that is, the semantics is holistic, not atomistic.

As an example, consider the incompatibility frame in which (1) A, B, C, and D are each pairwise incompatible with one another; (2) F, G, and H are also pairwise incompatible with one another; (3) P, Q, and R are an incompatible triad, as is R, S, and T (4) M is self-incompatible, (5) X is incompatible with every sentence listed so far, and finally (6) also includes the proposition Q.

5. From the incompatibility relations specified in an incompatibility frame, one may discern incompatibility sets for individual sentences or sets of sentences: sets of sets of sentences to which a given sentence (or set) is incompatible. We can actually think of such incompatibility sets to be a sentence’s semantic value, which we can then denote by the “stovepipe” (that is |φ| is to be understood referring to φ’s incompatibility set). Observe that this notion of a sentence’s semantic value is a much richer one than that provided by truth-functional semantics, in that one can do a lot more with a set of sets of sentences than one can do with a single, binary truth-value.

6. Thus, as we might seek to enrich a language through the addition of compound sentences formed by various logical operators, the project of computing those compound sentences’ semantic values will be that of showing how we may systematically compute their incompatibility sets from the original incompatibility frame (through, for instance, familiar set-theoretic operations).

7. Let’s think first about negation. Specifically, what sentences and sets of sentences ought to be incompatible with the negation of some statement? Clearly a statement and its negation are to be incompatible with one another, and so each need to belong to the other’s incompatibility set; indeed, if we think of all the sentences and sets of sentences incompatible with a statement, what they all have in common is that they preclude that statement. Observe that the statement to be negated is bound to be in the incompatibility set of anything with which it is incompatible. So, if we think that the basic function of affirming a negation of some statement is to rule out or deny the statement that is negated, then the set of sentences that are incompatible with the negation of a sentence may be understood as the intersection of the incompatibility sets of all the items in the original sentence’s incompatibility set. We call such a negation the original sentence’s minimal incompatible.

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1 The persistence constraint requires that for any set of sentences appearing in an incompatible frame, all supersets of that set must also appear in the frame. Basically, this constraint tells us that one may not repair an incompatible set of claims by adding more claims to it. Instead, one must retract (or subtract some sentence or sentences from that set in order to restore compatibility.
8. Similarly, think about the (sets of) statements that ought to be sufficient to rule out a disjunction of two propositions φ and ψ. Anything sufficient to rule out (φ v ψ) needs to be sufficient to rule out both φ and ψ. Thus the semantic value (or incompatibility set) of a disjunction may be defined as the intersection of the sets of sentences incompatible with each individual disjunction, meaning that anything incompatible with each of the disjuncts will also be incompatible with the disjunction. And the set of sentences incompatible with the conjunction of two propositions φ and ψ may be understood to be those incompatible with the set {φ, ψ}.

[At this point, you should ask yourself why it might not do to define the incompatibility set of a conjunction simply as the union of the incompatibility sets of its propositional components. To help focus your mind here, consider an incompatible triad of sentences, such as P, Q, and R above, or “This berry is red.”, “This berry is ripe.” and “This berry is a blackberry.” There is actually an important point here. The fact that the semantic value of a disjunction cannot simply be computed from the semantic values of its component disjuncts, but rather must look to wider features of the incompatibility frame, reminds us once again that the semantics here is holistic, rather than atomistic.]

9. Finally, one can fund from these basic incompatibility relations, a kind of incompatibility entailment (or way of unpacking the double turnstile), which follows from an idea formulated by the ancient stoics: one sentence or set of sentence incompatibility entails another just in case everything incompatible with the latter is also incompatible with the former. [Formally Φ |= Ψ just in case {X: X U Ψ ϵ INC} is a subset of {X: X U Φ ϵ INC}. For degenerate cases, Δ |= simply means that Δ is already part of the incompatibility frame, and so is self-incompatible.]

Thus one may not be committed to the former while also denying (or being precluded from) the latter (say, by being committed to something incompatible to it). Notice how this characterization of entailment rests on the pragmatic attitudes of affirming or denying sentences (or rather, the deontic statuses of commitment and preclusion that issue from such affirmations and denials). Indeed, we can show that this notion of incompatibility entailment accords fully with the notion of entailment that substitutes affirmation (or commitment) in for truth and denial (or preclusion) in for falsity. Here’s how: first, suppose that Γ incompatibility entails φ. That means that anything incompatible with φ must also be incompatible with Γ. But that means that any grounds for ruling out or denying φ would equally be sufficient for ruling out or denying Γ. So one could not coherently affirm everything in Γ while denying φ. Going the other direction, let's suppose, contrapositively, that Γ does not incompatibility entail φ. That means there must be something incompatible with φ that is not incompatible with Γ. And so one can coherently affirm whatever that might be alongside Γ, and thereby preclude oneself from (or deny) φ. Thus Γ could not entail φ when entailment is understood in terms of affirmation and denial (or commitment and preclusion).

10. Notice, then, that with our definition of negation, it turns out that anything in the incompatibility set of a negation of a sentence will also belong in the incompatibility set of anything incompatible to the negated sentence. That is, ~φ is to be understood as a sentence that is incompatibility entailed by any sentence incompatible with φ. That is in part what it means for the negation to be minimally incompatible with that which it negates. Similarly, we can easily verify that any disjunction will be incompatibility entailed by either of its disjuncts.
11. Incompatibility entailment also provides one with a rather natural and direct route for defining a kind of conditional. Roughly, \( \phi \rightarrow \psi \) just in case \( \phi \) incompatibility entails \( \psi \). Such a conditional differs from the ordinary truth-functional material conditional in that it avoids the various awkwardness and infelicity in having to hold vacuously true any conditional with either a false antecedent or a true consequent.

12. Exercises: For the incompatibility semantics described here,....

(1) Verify that the basic structural principles of entailment apply:

ASSUMPTIONS: For any sentence \( \phi \), \( \phi|=\phi \).

THINNING (or PERSISTENCE): If \( \Gamma|=\phi \), then \( \Gamma, \psi|=\phi \).

THE CUT: If \( \Gamma|=\phi \) and \( \phi, \Delta|=\psi \), then \( \Gamma, \Delta|=\psi \).

(2) Show that the “basic principle” for negation holds:

NEG: \( \Gamma|=\phi \) just in case \( \Gamma \) (or if and only if) \( \sim\phi |=. \)

(3) Show that the “basic principle” for disjunction holds:

DISJ: \( \Gamma, (\phi \lor \psi) |= \) just in case (or if and only if) \( \Gamma, \phi|= \) and \( \Gamma|=\psi \).

(4) Show that the “basic principle” for conjunction holds:

CONJ: \( \Gamma|= (\phi \land \psi) \) just in case \( \Gamma|=\phi \) and \( \Gamma|=\psi \).

[This is a bit trickier than the others. Here are some hints: Going from left to right will require you to appeal to the principle of persistence, while going from right to left will require you to invoke THE CUT.]

13. Answers:

(2) NEG (left to right): \( \Gamma|=\phi \), then \( \sim\phi |=. \)

Suppose \( \Gamma|=\phi \). So anything incompatible with \( \phi \) must also be incompatible with \( \Gamma \). But \( \sim\phi \) is defined as minimally incompatible with \( \Gamma \), and so must also be incompatible with \( \Gamma \), which is the same as saying \( \Gamma, \sim\phi |=. \)

NEG (right to left): \( \sim\phi |=, \) then \( \Gamma|=\phi \).

We start with the assumption that \( \Gamma, \sim\phi |=, \) meaning that \( \Gamma \) and \( \sim\phi \) are incompatible with each other. Now further suppose that some sentence or set of sentences \( \Theta \) is incompatible with \( \phi \). Then by the definition of \( \sim\phi \), \( \Theta \) must incompatibility entail \( \sim\phi \), and anything incompatible with \( \sim\phi \) must be
incompatible with Θ. But Γ is something incompatible with ~φ, and so it must be incompatible with Θ as well. So anything incompatible with φ must also be incompatible with Γ. That is, Γ|=φ.

(3) DISJ (left to right): If Γ, (φ v ψ) |=, then Γ, φ|= and Γ, ψ|=.

Now we start with the assumption that Γ, (φ v ψ) |=. That is to say, Γ is incompatible with (φ v ψ). Since the items incompatible with (φ v ψ) are defined to be those in the intersection of the items incompatible with φ and those incompatible with ψ, it follows then that Γ must be incompatible with both φ and ψ. And that is exactly what it means to say Γ, φ|= and Γ, ψ|=.

DISJ (right to left): If Γ, φ|= and Γ, ψ|=, then Γ, (φ v ψ) |=.

We begin with the assumption that Γ, φ|= and Γ, ψ|= (that is, both φ and ψ are incompatible with Γ). That means that Γ must fall in the intersection of the sets of those items incompatible φ and those items incompatible with ψ. And so by the definition of v, Γ must be incompatible with (φ v ψ).

(4) CONJ (left to right): If Γ |= (φ & ψ), then Γ |= φ and Γ|= ψ.

We start with the assumption that Γ |= (φ & ψ). And further suppose some sentence or set of sentences Θ is incompatible with φ. By the principle of persistence Θ would also have to be incompatible with {φ, ψ}. But by the definition of &, that means that Θ would also have to be incompatible with (φ & ψ). So (φ & ψ) incompatibility entails φ. Since anything incompatible with φ would be incompatible with (φ & ψ), and since by hypothesis, anything incompatible with (φ & ψ) is in turn incompatible with Γ, it follows that anything incompatible with φ must also be incompatible with Γ. That is, Γ |= φ. [An entirely parallel line of reasoning applies for ψ.]

CONJ (going right to left): If Γ |= φ and Γ|= ψ, then Γ |= (φ & ψ).

Let’s assume that Γ |= φ and Γ|= ψ. Now observe that by our definition of conjunction, anything incompatible with (φ & ψ) is that which is incompatible to the set {φ, ψ}. But that also would mean that φ, ψ |= (φ & ψ). Given our initial assumptions, we can apply THE CUT principle twice, and substitute Γ in for both φ and for ψ on the left hand side of that sequent. The resulting sequent is just what we want: Γ|= (φ & ψ).