Solutions for Assignment 4

1. (Data from Montgomery) Three brands of batteries are under study. It is suspected that the lives (in weeks) of the three brands are different. Five randomly selected batteries of each brand were tested. Note that the data are from a balanced design.

   (a) ANOVA Table:

Source of Variation	Degrees of Freedom	Sums of Squares	Mean Squares	F-ratio
treatment	$t-1$	SST=$r\sum_{j=1}^{t}(\bar{y}_{i.}-\bar{y}_{..})^2$	SST/t-1	MST/MSE
error	$t(r-1)$	SSE=$\sum_{i=1}^{r}\sum_{j=1}^{t}(y_{ij}-\bar{y}_{i.})^2$	SSE/t(r-1)	
total	$tr-1$	SSTot=$\sum_{i=1}^{r}\sum_{j=1}^{t}(y_{ij}-\bar{y}_{..})^2$		

```
> y <- matrix(c(100, 76, 108,
+  96, 80, 100,
+  92, 75, 96,
+  96, 84, 98,
+  92, 82, 100), 5,3, byrow=T)
>
> ## ANOVA table
> y.batt <- c(y[,1], y[,2], y[,3])
> x.batt <- rep(1:3, each=5)
> anova(lm(y.batt~as.factor(x.batt)))
Analysis of Variance Table

Response: y.batt
                  Df  Sum Sq Mean Sq F value
as.factor(x.batt)  2 1196.13  598.07  38.338
Residuals         12  187.20   15.60
```

   (b) Randomization test:

      i. For the test we have the following hypothesis:
         - H_0 (null hypothesis): There **are no** differences between the three types of batteries in regard to life-time.
         - H_1 (alternative hypothesis): There **are** differences between the three types of batteries in regard to life-time.

      ii. We will examine this question using the F-statistic as our test statistic for the randomization test. Thus we have:

$$F(\mathbf{Y_1}, \mathbf{Y_2}, \mathbf{Y_3}) \;=\; MST/MSE.$$

As above, we can use the `anova()` function in R to do the calculation with either the `lm()` or `aov()` functions. Based upon the p-value and setting $\alpha = 0.05$, we reject the null hypothesis since the p-value < 0.05.

```
> ## randominization test
> F.obs <- anova(lm(y.batt~as.factor(x.batt)))[1,4]
> F.null <- real()
> for(nsim in 1:5000){
+ x.sim <- sample(x.batt)
+ F.null[nsim] <- anova(lm(y.batt~as.factor(x.sim)))[1,4]
+ }
> p.value <- mean(F.null>=F.obs)
```

```
> p.value
[1] 4e-04
```

iii. Here we are assuming that the units are some type of shell for batteries and that the battery type is randomly assigned to these shells. In this case, the battery type includes not just the design of the battery, but also the manufacturing process.

(c) Test based upon the data being samples from a population:

i. Here the null and alternative hypotheses are based on population parameters. In order to conduct the test, we make the following assumptions:

$$\begin{aligned} y_{ij} &= \mu_j + \epsilon_{ij}, \\ \epsilon_{ij} &\sim \text{i.i.d. normal}(0, \sigma). \end{aligned}$$

ii. To conduct the test, we can determine the p-value:

$$\begin{aligned} \text{p-value} &= P(F_{t-1,t(r-1)} > F_{obs}) \\ &= \texttt{1-pf(38.338, 3-1,15-3)} \\ &= 6.141167e-06. \end{aligned}$$

iii. Since the p-value < 0.05 we reject the null hypothesis.

iv. The formula for the 95% confidence intervals are:

$$\begin{aligned} \bar{y}_{i.} &\pm SE(\bar{y}_{i.}) \times t_{1-\alpha/2,N-t}, \\ SE(\bar{y}_{i.}) &= \sqrt{s^2/r_i} = \sqrt{s^2/5}, \\ s^2 &= MSE = 15.60. \end{aligned}$$

Since we have a balanced design the SE($\bar{y}_{i.}$) is the same for every battery type.

Battery Type	$\hat{\mu}_{batt}$	r	SE($\hat{\mu}_{batt}$)	95% CI
1	95.2	5	1.766	(91.351,99.049)
2	79.4	5	1.766	(75.551, 83.249)
3	100.4	5	1.766	(96.551, 104.249)

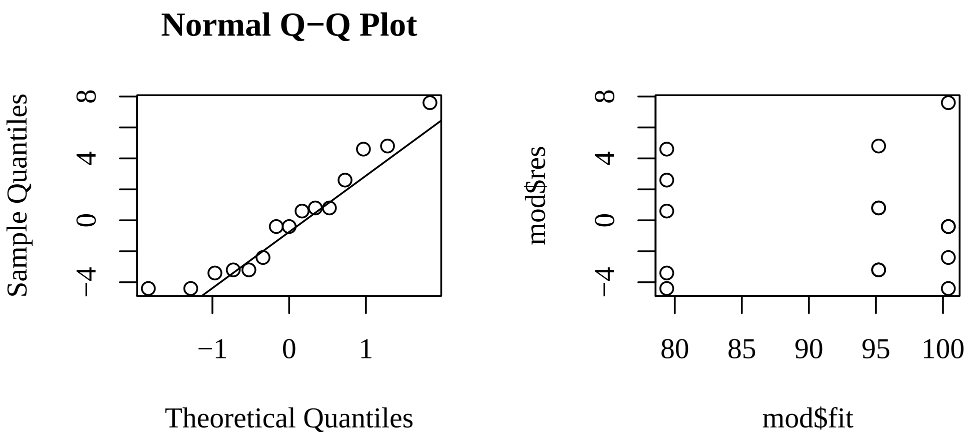
v. 95% CIs of the treatment comparisons with Bonferroni correction:

$$\begin{aligned} \hat{C} &\pm SE(\hat{C}) \times t_{1-(\alpha/3)/2,N-t}, \\ \hat{C} &= \sum_{i=1}^{t} k_i \bar{y}_{i.}, \\ SE(\hat{C}) &= s\sqrt{\sum_{i=1}^{t} k_i^2/r_i}. \end{aligned}$$

Comparison	k	$\hat{C}$	SE($\hat{C}$)	95% CI
$\mu_1 - \mu_2$	(1,-1,0)	15.8	2.498	(8.857, 22.743)
$\mu_1 - \mu_3$	(1,0,-1)	-5.2	2.498	(-12.143, 1.743)
$\mu_2 - \mu_3$	(0,1,-1)	-21.0	2.498	(-27.943, -14.057)

From the table, there is a statistical difference between the first and second type and between the second and third type. But we can not say that there is a difference between the first and third type based upon $\alpha = 0.05$, since the interval contains zero.

(d) Diagnostic plots:



From the Q-Q plot, there does not seem to be a strong departure from normality. From the fitted vs residual plot, the values seem somewhat random and the variance does not seem to have increase with the mean. But perhaps there is an outlier. Finally, $s^2_{largest}/s^2_{smallest} < 3$ so we should not worry.

(e) Power calculation($\sum \tau_i^2/t = 9, \sigma^2 = 20$, and $\alpha = 0.05$):

```
> ## power
> pow <- function(r){
+ alpha <- 0.05
+ sigma.sq <- 20
+ t <- 3
+ N <- r*t
+ lambda <- (r*t*9)/sigma.sq
+ pow <-1-pf(qf(1-alpha, t-1, N-t), t-1, N-t, ncp=lambda)
+ return(pow)
+ }
>
> pow(5)
[1] 0.5234859
> pow(7)
[1] 0.7158705
> pow(8)
[1] 0.7864282
> pow(9)
[1] 0.8418769
```