Lagrange Multiplier Based Domain Decomposition Method with Non-matching Grids for Time-dependent Equations

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Abstract

Our intention in this thesis is to study one kind of non-overlapping domain decomposed finite element method systematically — A Lagrange multiplier based domain decomposition method with non-matching grids for time-dependent problem. This method not only refers to methods defined on a decomposition of domain consisting of a collection of mutually disjoint subdomains, but also allows discontinuity of the interior variables across the boundary of the subdomains. We applied the method to parabolic equation, proposed its semi-discrete and fully discrete finite element approximated schemes of Lagrange multiplier based DDM with non-matching grids, studied their finite element solutions' existence and uniqueness, and obtained their optimal error estimate for H^1-norm and L^2-norm.

1. Introduction

We consider the following model problem

$$\begin{cases} \frac{\partial u}{\partial t} - \triangle u = f, & (x,t) \in \Omega \times (0,T], \\ u = 0, & (x,t) \in \partial\Omega \times (0,T], \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset R^2$ is a quasi-uniform polygonal domain with size d,$\partial\Omega$ is its boundary with piecewise *Lipschitz* smooth,$T > 0$ is a constant, $f \in L^2(0,T;L^2(\Omega))$.

2. Domain Decomposition and Finite Element Space

We first decompose the domain Ω into subdomains Ω_i,and then sub-divide every subdomain Ω_i into finite elements. We make the following assumption:

A1. Ω is a polygonal domain.

A2. Ω is decomposed into quasi-uniform subdomains $\Omega_i(i = 1,2,\cdots,n_d)$ with size d. It means that there is a positive constant c independent of d such that every Ω_i contains a disk of diameter cd and is contained in a disk of diameter d and every side of Ω_i is greater than $c_1 d$,c_1 independent of d.

A3. Every subdomain Ω_i is subdivided in the above sense into quasi-uniform triangular or quadrilateral elements with size h_i. Let $\Omega_i^{h_i}$ be the set of all elements in Ω_i,$\Omega^h = \bigcup\limits_i \Omega_i^{h_i}$.

Denote the vertices of Ω_i by γ_j(in some order). Γ_{ij} is the side of Ω_i with the vertices γ_i and γ_j. $\Gamma = \bigcup\limits_{i=1}^{n_d} \partial\Omega_i/\partial\Omega$.

We now construct the finite element space.

Let $S_{h_i}(\Omega_i)$ be the space of piecewise m_i-th degree continuous polynomials defined on $\Omega_i^{h_i}$. Define the space $S_h(\Omega)$,consisting of piecewise polynomials on Ω^h,as follows:

$$S_h(\Omega) = \{f | f \text{ is the piecewise } m_i\text{-th degree continuous polynomial on } \Omega_i^{h_i}\}.$$

Notice that the function in $S_h(\Omega)$ is only continuous in the interior of every subdomain but may be discontinuous on $\partial\Omega_i$.

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Let

$S_n(\Gamma_{ij}) = \{f(x)|f(x)$ is an n-th order polynomial on $\Gamma_{ij}\}$,

$S_n(\Gamma) = \{\lambda(\lambda)|\lambda|_{\Gamma_{ij}} \in S_n(\Gamma_{ij})\}$,

$S_n(\partial\Omega_i) = \{\lambda(\lambda)|\lambda|_{\Gamma_{ij}} \in S_n(\Gamma_{ij}), \Gamma_{ij} \subset \partial\Omega_i\}$.

Notice that the function in $S_n(\Gamma)$ may be discontinuous at the vertices of the subdomains.

Define the finite element space by

$$S_{h\times n} = S_h(\Omega) \times S_n(\Gamma),$$

$(u,\lambda) \in S_{h\times n}$ if and only if $(u,\lambda) \in S_h(\Omega) \times S_n(\Gamma)$. Obviously,$S_n(\Gamma) = \{\lambda|(u,\lambda) \in S_{h\times n}\}$.

For the parameters h_i, d, n, we always assume that

$$0 < h_i < d, i = 1, 2, \cdots, n_d,$$

and

$$\lim_{h\to 0} \frac{n^2 h}{d} = 0,$$

where $h = \max_i\{h_i\}$.

3. Domain Decomposed Semi-discrete Finite element Method with nonmatching Grids

3.1. Semi-discrete Finite Element Approximated Scheme

After we decompose Ω into a set of subdomain $\bigcup_{i=1}^{n_d} \Omega_i$, the weak form of (1.1) is:

To find $(u,\lambda) : [0,T] \to H(\Omega) \times H^{-\frac{1}{2}}(\Gamma)$,satisfying

$$\begin{cases} \sum_i\{(u_t, v)_{\Omega_i} + (\nabla u, \nabla v)_{\Omega_i} - \langle v, \lambda\rangle_{\partial\Omega_i}\} \\ = \sum_i (f, v)_{\Omega_i}, \\ \sum_i \langle u, \mu\rangle_{\partial\Omega_i} = 0, \qquad \forall (v,\mu) \in H(\Omega) \times H^{-\frac{1}{2}}(\Gamma), \end{cases} \tag{3.1}$$

where $H(\Omega) = H^1(\Omega_1) \oplus \ldots \oplus H^1(\Omega_{n_d})$.

We now construct the semi-discrete finite element approximate scheme. To find $(U, \Lambda) : [0,T] \to S_{h\times n}$, such that

$$\begin{cases} \sum_i\{(U_t, v)_{\Omega_i} + (\nabla U, \nabla v)_{\Omega_i} - \langle v, \Lambda\rangle_{\partial\Omega_i}\} \\ \qquad = \sum_i (f, v)_{\Omega_i}, \\ \sum_i \langle U, \mu\rangle_{\partial\Omega_i} = 0, \\ U(x, 0) = \widetilde{u}_0(x), \qquad \forall (v,\mu) \in S_{h\times n}, \end{cases} \tag{3.2}$$

where $\widetilde{u}_0(x)$ is defined as following (3.3).

From [1] and *Ladyzhenskaya-Babuška-Brezzi (LBB)* condition,it is easy to know finite element solution of problem (2) is existent and unique.

3.2. Error Estimation

To estimate the error between (3.2) and (3.1), we introduce a kind of projection about $(u, \frac{\partial u}{\partial n})$, named H^1 projection, defined by finding $(\widetilde{u}, \widetilde{\lambda}) : [0,T] \to S_{h\times n}$, such that:

$$\begin{cases} \sum_i\{(\nabla(\widetilde{u} - u), \nabla v)_{\Omega_i} + (\widetilde{u} - u, v)_{\Omega_i} \\ \qquad -\langle v, \widetilde{\lambda} - \frac{\partial u}{\partial n}\rangle_{\partial\Omega_i}\} = 0, \\ \sum_i \langle \widetilde{u} - u, \mu\rangle_{\partial\Omega_i} = 0, \qquad \forall (v,\mu) \in S_{h\times n}. \end{cases} \tag{3.3}$$

It follows from [1] that $(\widetilde{u}, \widetilde{\lambda})$ is existent and unique. The error estimation between $(\widetilde{u}, \widetilde{\lambda})$ and $(u, \frac{\partial u}{\partial n})$ satisfy the lemma as follow:

We analyzed the error estimation between (3.2) and (3.1), and obtain the following theorem

Theorem 1 *Let $h < d, m \le n, k \ge 2m, u$ is the mathematical solution of problem (1.1), and $u \in W^{1,2}(0,T; H^{k+1}(\Omega_i)), i = 1, \ldots, n_d$, (U, Λ) is the finite element solution of (2), then*

$$\begin{aligned} &(\sum_i(\|U - u\|^2_{L^\infty(L^2(\Omega_i))} + \|(U-u)_t\|^2_{L^2(L^2(\Omega_i))} \\ &+(h + n^{-1}d^{\frac{1}{2}})^2\|U - u\|^2_{L^\infty(H^1(\Omega_i))} \\ &+(h + n^{-1}d^{\frac{1}{2}})^2\|\Lambda - \tfrac{\partial u}{\partial n}\|^2_{L^2(H^{-\frac{1}{2}}(\partial\Omega_i))}))^{\frac{1}{2}} \\ &\lesssim h^m(h + n^{-1}d^{\frac{1}{2}}) \cdot (\sum_i(\|u\|^2_{L^2(H^{m+1}(\Omega_i))} \\ &+\|u_t\|^2_{L^2(H^{m+1}(\Omega_i))} + \|u\|^2_{L^2(H^{k+\frac{1}{2}}(\partial\Omega_i))} \\ &+\|u_t\|^2_{L^2(H^{k+\frac{1}{2}}(\partial\Omega_i))}))^{\frac{1}{2}}. \end{aligned}$$

4. Domain Decomposed Full-discrete Finite element Method with nonmatching Grids

4.1. Full discrete Finite Element Approximated Scheme

First of all,we decomposed equally the time domain $[0,T]$ into N parts: $0 = t_0 < t_1 < \ldots < t_N = T, \tau = t_{j+1} - t_j = \frac{T}{N}$, $t_j = j\tau$. We introduce some notations

$$v^j = v(x, t_j), j = 0, 1, \cdots, N;$$

$$v^{j+\frac{1}{2}} = \frac{1}{2}(v^{j+1} + v^j), \partial_t v^{j+\frac{1}{2}} = \frac{1}{\tau}(v^{j+1} - v^j),$$

$$j = 0, 1, \cdots, N-1.$$

The full-discrete approximation for domain decomposition method with non-matching grids is defined by the sequence, find $(\{U^j\}_{j=0}^N, \{\Lambda^j\}_{j=0}^N) : [0,T] \to S_{h\times n}$,such that

$$\begin{cases} \sum_i \{(\partial_t U^{j+\frac{1}{2}}, v)_{\Omega_i} + (\nabla U^{j+\frac{1}{2}}, \nabla v)_{\Omega_i} \\ -\langle v, \Lambda^{j+\frac{1}{2}}\rangle_{\partial\Omega_i}\} \\ \qquad = \sum_i (f^{j+\frac{1}{2}}, v)_{\Omega_i}, \\ \sum_i \langle U^{j+1}, \mu\rangle_{\partial\Omega_i} = 0, \\ U^0 = \widetilde{u}_0(x), \forall (v,\mu) \in S_{h\times n},\ j = 0, 1, \ldots, N-1, \end{cases} \tag{4.1}$$

where $\widetilde{u}_0(x)$ is defined by (3.3).

4.2. Algorithm's description

1.Guess U^0 initially, the j step of time = 0, t=0.

2. Given j=j+1,$t = t + \tau$ until $j = N, t = T$, we can compute respective stiffness and vector on each time step: M_i, S_i, L_i and F_i, which is local boundary stiffness matrix $L_i^T(M_i + \frac{\tau}{2}S_i)^{-1}L_i$ and local equivalent load $-L_i^T(M_i + \frac{\tau}{2}S_i)^{-1}(\frac{\tau}{2})^{-1}F_i$. The total boundary stiffness matrix S and the total equivalent load F can be obtained by summation over them.

3.Solve $S\Lambda^{j+1} = F$ by iteration,where Λ^{j+1} is the unknown on the boundary, consisting of Λ_i^{j+1},$(i = 1, \cdots, n_d)$.

4.Find U^{j+1} in (4.4), $(i = 1, \ldots, n_d)$.

5.Go to the second step and continue the time step's iteration.

In implementation, we only need the local boundary stiffness matrix and the local equivalent load since we solve $S\Lambda^{j+1} = F$ by iteration. The algorithm is suitable to parallel computing.

4.3. Error Estimation

We list out the theorem of error estimate between (4.1) and (3.1), where we use the H^1 projection $(\widetilde{u}, \widetilde{\lambda})$ which was based on *Lagrange* multiplier of DDM and has been defined in §2. Notations ξ, η, ζ, θ are the same as that in §2.

Theorem 2 *If* $u_{t^3} \in L^2(0,T; L^2(\Omega_i)), i = 1, \cdots, n_d$, $(\{U^j\}_{j=0}^N, \{\Lambda^j\}_{j=0}^N)$ *is the finite element solution of (4.1), then under the conditions of Theorem1, we have*

$$\begin{aligned} &\sup_{1\le M\le N} \{\|U^M - u^M\|_{0,\Omega} \\ &+(\tau \sum_{j=0}^{M-1} \|\partial_t(U^{j+\frac{1}{2}} - u^{j+\frac{1}{2}})\|_{0,\Omega}^2)^{\frac{1}{2}} \\ &+(h + n^{-1}d^{\frac{1}{2}})(\sum_i \|U^M - u^M\|_{1,\Omega_i}^2)^{\frac{1}{2}} \\ &+(h + n^{-1}d^{\frac{1}{2}})(\tau \sum_{j=0}^{M-1} \sum_i |\Lambda^{j+\frac{1}{2}} - (\tfrac{\partial u}{\partial n})^{j+\frac{1}{2}}|_{-\frac{1}{2},\partial\Omega_i}^2)^{\frac{1}{2}}\} \\ &\lesssim \{h^m(h + n^{-1}d^{\frac{1}{2}}) + \tau^2\}\{(\sum_i \|u\|_{L^2(H^{m+1}(\Omega_i))}^2)^{\frac{1}{2}} \\ &+(\sum_i \|u\|_{L^2(H^{k+\frac{1}{2}}(\partial\Omega_i))}^2)^{\frac{1}{2}} + (\sum_i \|u_t\|_{L^2(H^{m+1}(\Omega_i))}^2)^{\frac{1}{2}} \\ &+(\sum_i \|u_t\|_{L^2(H^{k+\frac{1}{2}}(\partial\Omega_i))}^2)^{\frac{1}{2}} + (\sum_i \|u_{t^3}\|_{L^2(L^2(\Omega_i))}^2)^{\frac{1}{2}}\}. \end{aligned}$$

References

[1] Guoping Liang, Jiangheng He,Incompatible Domain Decompositon Lagrangian Multiplier. *J. Comp. Math.* , No.**2**(1992), pp207–215.

[2] Howard Swann, On the use of Lagrange multipliers in domain decomposition for solving elliptic problems. *Math. Comp.* , **60**(1993), pp49–78.

[3] C.Farhat, J.Mandel, & F.X.Roux, Optimal convergence properties of the FETI domain decomposition method. *Comput. Methods Appl. Mech. Eng.*, **115**(1994), pp367–388.

[4] J.Mandel, R.Tezaur, Convergence of a substructuring method with Lagrange multipliers. *UCD/CCM Report 33*, University of Colorado at Denver,1994.

[5] P.Le,Tallec,T.Sassi, & M.Vidrascu, Three-dimensional domain decomposition methods with non-matching grids and unstructured coarse solvers. *Contemporary Mathematics*, **180**(1994), pp61–74.

[6] C.Farhat, J.Mandel, & Po-shu Chen, A scalable Lagrange multiplier based domain decomposition method for time-dependent problems. *Int. J. Numer. Methods Eng.*, **38**(1995), pp3831–3853.

[7] C.Farhat, F.X.Roux, Implicit parallel processing in structural mechanics. *Comp. Mech. Adv.*, **2**(1994), pp1–124.

[8] Guoping Liang, Ping Liang, Domain Decomposition Method of Hybrid Finite Element, *J. Comp. Math.* ,No.**3**(1989), pp363–370.