Existence of Solutions

(E) \( \dot{x} = f(t, x) \)

Problem: To find a differentiable function \( \varphi \) defined on a real \( t \) interval \( I \) such that

(i) \((t, \varphi(t)) \in D \quad (t \in I)\) \[ D \text{ is a domain, i.e., an open and connected set in the real } (t,x) \text{ plane.} \]

(ii) \( \dot{\varphi}(t) = f(t, \varphi(t)) \quad (t \in I) \)

If such an interval \( I \) exists and function \( \varphi \) exists, then \( \varphi \) is called a solution of the differential equation (E) on \( I \). If \( \varphi \) is a sol'n of (E) then \( \varphi \in C^1_I \).

(IVP) Initial Value Problem. To find an interval \( I \) containing \( 2 \) and a solution \( \varphi \) of (E) on \( I \) satisfying \( \varphi(2) = 2 \).

We can solve IVP by solving (by Fundamental Theorem of Calculus)

\[ \varphi(t) = 2 + \int_2^t f(s, \varphi(s)) \, ds \quad (t \in I) \]

If \( f \) is continuous on \( I \), then (E) \( \equiv \) (IE).

Differential Equations with Discontinuous Right Hand Side

Case 1: \( \dot{x} = \text{sgn}(x) \)

For \( t < 0 \), \( x = -1 \), \( x(t) = -t + c_1 \)

For \( t > 0 \), \( x = 1 \), \( x(t) = t + c_2 \)

For solution continuity at \( t = 0 \)

\[ x(0) = \lim_{t \to 0} (-t + c_1) = x(t + c_1) \]

\[ t \to 0 \]

\[ x(0) = c_1 = c_2 \]

\[ x(t) = t + c \]

For \( t = 0 \), \( \dot{x}(t) \) doesn't exist.
\[ x = 1 - 2 \sin(x) \]

For \( x < 0 \), \( x = 3 \), \( x(t) = 3t + c_1 \)

For \( x > 0 \), \( x = -1 \), \( x(t) = -t + c_2 \)

If we set \( x = 0 \) as a solution after a trajectory hits zero, then \( 1 - 2 \sin(0) = 1 \neq 0 \)

- For cases where \( f(t,x) \) is continuous in \( x \) and discontinuous in \( t \), we can just use mathematical argument to generalize the solution. (Controversy: solution)

- For \( f(t,x) \) discontinuous in \( t \), needs some limiting for using some physical meaning into account.

- Generalization of solution concept (Requirement):

  1. The solution concept should be valid for \( f(t,x) \) with \( f \in C \).
  2. For \( x = f(t) \), it should turn out to be \( x(t) = \int f(t) \, dt + c \) only.
  3. Under any initial data \( x(0) = x_0 \) in a given region of the solution must exist (at least for \( t \geq 0 \)) and continue to the boundary of this region or to infinity \( c \in (h,x) \to \infty \).
  4. The def of solution should work for many physical systems, moreover...
  5. The limit of a uniformly convergent sequence of solutions must be a solution.
  6. Under the commonly used change of variables a role must remain a role.

References:

1. Differential Equation with Discontinuous Right-hand Sides, A. F. Filippov
Caratheodory conditions: In the domain $D$ of the $(t,x)$ space,

1. $f(t,x)$ be defined and continuous in $x$ for almost all $t$;
   (except in a set of measure 0).

2. $f(x,t)$ be measurable in $t$ for each $x$.

3. $|f(t,x)| \leq m(t)$, $m(t)$ is a summable
   function on each finite interval $t$ in $D$ if $t$ is not bounded in $D$.
   or $m(t)$ is a Lebesgue integrable function.

$L = f(t,x) \in \mathbb{R}^n$, and $f(t,x)$ satisfies 1-3

is called the Caratheodory Equation.

(deduction of the Caratheodory Equation)

Problem: To find an absolutely continuous $\phi$, defined on
a real $t$ interval $I$, such that $\phi(t) \in \mathbb{R}^n$ for $t \in I$.

(i) $\phi(t)$ is a continuous function defined on $I$.

(ii) $\phi(t) = f(t, \phi(t))$, for all $t \in I$ except on a
    set of Lebesgue measure zero.

Discontinuous Right Hand Side:

Defn: Piecewise continuous function in a finite domain $G$ of an
$(n+1)$ dimensional $(t,x)$ space if the domain $G$ consists of
a finite # of domains $G_i (i = 1, \ldots, l)$ in each of which
the function is continuous up to the boundary, and of a
set $M$ of measure zero which consist of boundary point
of these domains.

Continuous up to the boundary: means the $f(t)$ has a
finite limit at the boundary.
Most common example of \( M \) (finite \# of hypersurfaces or manifolds).

In an \( n \)-dimensional space, a set \( S \) is called a \( k \)-dimensional hypersurface or manifold if \( \forall \) \( x \) in the neighborhood of each of its points \( a \), all \( k \) coordinates of the points of the set \( S \) are continuous functions of some \( k \) coordinates varying over a certain \( k \)-dimensional domain \( C^k(a) \). E.g.

\[
x_i = \phi_i(x_1, \ldots, x_k) \in C^k, i = 1, \ldots, m, \quad (x_i, \ldots, x_k) \in C^k(a)
\]

Hyperplanes of class \( C^0 \) indicate \( M \), with \( \phi_i \) being \( C^0 \).

If \( \phi_i \) are called smooth, \( \phi_i \) being \( C^\infty \).

One-dimensional example:

Take \( f_+(x_0) \) and \( f_-(x_0) \)

(i) \( f_+ \) and \( f_- \) point towards \( S_+ \)

Contradiction interpretation is sufficient.

(ii) \( f_+ \) and \( f_- \) towards \( S_- \)

Contradiction is sufficient.

(iii) \( x_0 \) not to be loaded.

For \( x_0 \) as the initial condition, two solutions possible.

(iv) \( f_+ \) points inside \( S_+ \) and \( f_- \) outside \( S_- \)

Contradiction doesn't work.

Solution concept: \( \dot{x} = f(t, x) \) \( \ominus 1 \)

where \( f \in \text{PWC in domain } \Omega, \ x \in \mathbb{R}^n \), \( \Omega \) a set of (of measure 0)

of points of discontinuities of \( f \).

Define a set-valued function \( F(t, x) \) as follows (on domain \( \Omega \),

If \( f \) at a point \( (t, x) \), \( f \) is continuous, then \( F(t, x) = \{ f(t, x) \} \)

For other \( \tilde{x} \) of \( \Omega \), create \( F(t, x) \) from \( f(t, x) \).
Solution of 1. in terms of
\[ \dot{z} \in F(z(t)) \text{ almost everywhere.} \]
Absolutely continuous vector valued for \( z(t) \) defined on an interval I.

\[ \dot{z} \in F(z(t)) \]

1. Convex Hull

\[ \text{Convex Hull} \quad x_0 \quad x^+ \quad x^- \quad \text{convex hull} \]

\[ 0 = \alpha b^+ + (1-\alpha)b^- \text{ on the tangent space of } S_0. \]

2. Equivalent Control Method: \( f \) of the form \( f(x, u(x)) \)

\[ u(x) \text{ is single valued on } S^+ \cup S^- \text{ but has a range of values } U(x) \text{ for } x \in S_0. \]

We seek \( u_c(x) \) for \( x \in S_0 \) s.t. \( f(x, u_c(x)) \) is tangent to \( S_0 \) and \( u_c(x) \in U(x) \).

3. \( \dot{x} = f(x, u_c(x)) \), \( u_c(x) \in U(x) \) \{ \( = U(x) \), \( x \in S^+ \cup S^- \) \}

\[ F(x_0); \text{smallest convex set containing } \{(f(x_0, u) \mid u \in U(x))\} \]

\[ \text{use } \dot{x} \in F(x(t)) \]

If \( f(x, u) \) depends affinely on \( u \) and \( U(x_0) \) is \( [u_-, u_+] \)

\[ \dot{u}_+ = \frac{d}{dx} x_{s+} \quad \quad \dot{u}_- = \frac{d}{dx} x_{s-} \quad x \to x_0 \]

\[ \text{eq.} \quad \text{are equivalent:} \]

Define \[ g_u(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \\ \epsilon & x = 0 \end{cases} \]

\[ f(x, u) = \frac{1}{2} (1+u) b^+ (x) + \frac{1}{2} (1-u) b^- (x) \]

\[ \dot{x}_1 = \cos(\theta u), \quad \dot{x}_2 = -\sin(\theta u), \quad y = x_2, \quad u = \sin y \]

or \[ \dot{x}_1 = \cos(\theta \sin(x_2)), \quad \dot{x}_2 = -\sin(\theta \sin(x_2)) \]

for \( x_2 > 0 \), \( \dot{x}_1 = \cos \theta \), \( \dot{x}_2 = -\sin \theta \)

for \( x_2 < 0 \), \( \dot{x}_1 = \cos \theta \), \( \dot{x}_2 = \sin \theta \)

Sliding mode on \( x_1 = 0 \)

for \( \dot{x}_1 = 0 \), \( \dot{x}_2 = 0 \)

\[ u_c = 0 \Rightarrow \dot{x}_1 = 1 \]
\[
\begin{bmatrix}
\cos \theta \\
-\sin \theta
\end{bmatrix}, \quad \beta = \begin{bmatrix}
\cos \theta \\
\omega^2 \beta
\end{bmatrix}
\]

We need \( \beta_0 = \alpha \beta^+ + (1-\alpha) \beta^- \), \( \alpha + \beta \neq 0 \)

Take \( \beta \approx \frac{1}{2} \)

and we get \( \dot{\beta} = \cos \theta \)

Mind definition gives
\( \dot{\beta} \in [\cos \theta, 1] \)

**Approximate models**

1. Continuous
   \( \dot{x}_1 = \cos(\theta \tanh(x_2/\epsilon)), \quad \dot{x}_2 = -\sin(\theta \tanh(x_2/\epsilon)) \)

   Outside a narrow band around switching surface, similar trajectories. On \( S_0 \), \( x_1(+) + C \cdot x_2(+) = 0 \)

   \( \implies \dot{x}_1 = 1 \) same as equivalent control.

2. Approximated
   \( I = \sin h, \quad I_{x_1} = \cos \theta + C \cdot \sin \theta \)

   \( \dot{x}_1 = \cos \theta \) (simpler convex definition)

3. \( \dot{x}_1 = \cos \theta, \quad \dot{x}_2 = -\tanh(x_2(\epsilon)) \sin \theta \)

   \( \epsilon \approx \epsilon(\theta) > 0 \)

4. \( \dot{x}_1 = \cos \theta, \quad \dot{x}_2 = -\sin \theta, \quad \epsilon \approx \epsilon(\theta) > 0 \)

   Smooth approximations of \( \beta \) is slightly.

   Also gives \( \dot{x}_1 = \cos \theta \) on \( S_0 \)
\[ x_1(t) = -x_1(t) + x_2(t) - u(t) \]
\[ x_2(t) = 2x_2(t) (u^2(t) - u(t) - 1) \]
\[ u(t) = \text{sfun} x_1(t) \]

Sliding mode at \( x_1 = 0 \), \(-1 \leq x_2 \leq 1\).

Convex def. gives the sliding dynamics as \( \dot{x}_2 = -2x_2^2 \rightarrow \) has an unstable equil. at \((0,0)\)

Equivalent center
\[ \dot{x}_2 = 2x_2(x_2^2 - x_2 - 1) \rightarrow 2 \text{ equilibria} \]

(\( \frac{1}{2} - \frac{1}{4}\sqrt{5} \)) is unstable
\( x(0) \) is stable

Simplified
\[ u = \tanh(x/\delta) \]

EFF (evelflow formula) for eq. controllability
\[ \dot{x} = f(x,u), \quad y = g(x) \]
\[ \begin{cases} 
  y > 0, & u = u_+(x) \\
  y < 0, & u = u_-(x) \\
  y = 0, & u_-(x) \leq u \leq u_+(x) 
\end{cases} \]

For convex
\[ \dot{x} = \frac{1}{2}(1+u) f_+(x) + \frac{1}{2}(1-u) f_-(x), \quad y = g(x) \]
\[ \begin{cases} 
  y > 0, & u > 1 \\
  y < 0, & u < 1 \\
  y = 0, & -1 \leq u \leq 1 
\end{cases} \]

... these mode...