STABILITY

Stability is concerned with behavior of system around equilibrium point. Sometimes, we need to study stability about a nominal trajectory. We can convert this to equilibrium stability problem: $\dot{x} = f(t, x), \quad x(0) = x_0$

Let $x(0)$ be a solution of this.

Let $x(t) = x_0 + \delta x_0$

Now $\dot{x}^* = \delta f(x^*), \quad x_0 = x_0$

and $\dot{x} = \delta f(x), \quad x(0) = x_0 + \delta x_0$

Take $e(t) = x(t) - x^*(t)$

Then $\dot{e}(t) = \delta f(x^* + e, t) - \delta f(x^*(t), t) = g(e, t)$

where $e = 0$ is the equilibrium.

STABILITY FOR AUTONOMOUS SYSTEMS

STABILITY: The equilibrium point $x = 0$ is stable if

$\forall R > 0, \exists \delta > 0, \|x(0)\| < \delta \Rightarrow \forall t > 0, \|x(t)\| < R$

otherwise unstable.

Example: Van der Pol Oscillator

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + (1 - x_1^2)x_2
\end{align*}
\]

\[
\text{limit cycle}
\]

\[
\text{trajectories starting from nonzero IC,}
\]

\[
\text{approach limit cycle. According to definition, unstable (although bounded). In linear system, unstable \Rightarrow "blowing up".}
\]

ASYMPTOTIC STABILITY: An equilibrium point $0$ for $\dot{x} = f(x)$

is asymptotically stable if

1) it is stable

2) $\exists \delta > 0, \forall \epsilon > 0, \|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, t \to \infty$

Be: Ball of attraction (or domain of attraction)

Equilibrium point stable but not asymptotically is called marginally stable.
STABILITY

AUTONOMOUS SYSTEMS

\[ \dot{x} = f(x) \]

Equilibrium point at \( x^* \), satisfying \( f(x^*) = 0 \)

We want to study nonlinear systems \( \dot{x} = f(x) \) with 0 as equilibrium point.

Let the equilibrium point of \( \dot{x} = f(x) \) be at \( x^* \) (which is \( \neq 0 \)). Then define a new state variable \( y = x - x^* \).

Then \( \dot{y} = \dot{x} = f(x) = f(y + x^*) \)

\[ \therefore \text{ the transformed system representation is} \]
\[ \dot{y} = f(y + x^*) \text{ which has equilibrium point } y^* = 0. \]
Necessity of having condition 1.
Example of a nonlinear system (solved by Vinograd)

\[ x(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \]

**Exponentially Stable** equilibrium point 0 of \( x = f(x) \)
is exponentially stable, if \( \exists \quad \lambda \quad \text{and} \quad A > 0 \quad \text{such that} \quad \forall t > 0, \quad \| x(t) \| \leq \lambda \| x(0) \| e^{-\lambda t} \)

Some ball be around the origin.

**Global Stability** (asymptotic or exponential) if
asymptotic (or exponential) stability holds for any
initial state, then global asymptotic (or exponential)
stability. (Also called asymptotic (or exponential)
stability in the large)

**Linearization**

\[ x = f(x) \quad \text{with} \quad x = 0 \quad \text{as equilibrium} \]

\[ \dot{x} = \left( \frac{\partial f}{\partial x} \right)_{x=0} + 0 \cdot h_0(t) \quad (x) \]

\[ \dot{x} = Ax, \quad \text{with} \quad A = \left( \frac{\partial f}{\partial x} \right)_{x=0} \quad (Jacobian of f) \]

**Lyapunov's Linearization Method** (Lyapunov's First Method)

- If eigenvalues of linearized system are in left half plane, then
  the actual nonlinear system, locally asymptotically stable.
- If at least one eigenvalue of the linearized system has
  a real part \( > 0 \) \(
  \text{nonlinear system unstable.}
- If linearized system marginally stable (all eigenvalues
  in left half plane except one or more on jω axis) \Rightarrow
  no conclusions about nonlinear system.
SOME PRELIMINARIES FOR LYAPUNOV DIRECT METHOD

POSITIVE DEFINITE FUNCTION: A scalar function $V(x)$ is locally +ve definite if $V(0) = 0$ and in a ball $B_R$
$$x \neq 0 \Rightarrow V(x) > 0$$
If $V(0) = 0$ and $V(x) \geq 0$ for all $x \in \mathbb{R}^n$ except $x = 0$, then $V(x)$ is globally +ve definite.

NEGATIVE DEFINITE if $-V(x)$ is +ve definite.

+ve semi definite: If $V(0) = 0$ and $V(x) \geq 0$ for $x \neq 0$.

-ve semi definite: If $-V(x)$ is +ve semi definite.

Derivative of $V$ w.r.t. $x$
$$\dot{V} = \frac{dV(x)}{dt} = \frac{\partial V}{\partial x} \cdot \dot{x} = \frac{\partial V}{\partial x} f(x)$$

Lyapunov function: $\dot{x} = f(x)$

Let 1) $V(x)$ be +ve definite in $B_R$
2) $\frac{\partial V}{\partial x}$ exists and be continuous in $B_R$
3) $\dot{V}(x) \leq 0$

Then, $V(x)$ is a Lyapunov function for $\dot{x} = f(x)$

LYAPUNOV THEOREM FOR LOCAL STABILITY

Let $x = 0$ be equilibrium point for $\dot{x} = f(x)$ and $D \subset \mathbb{R}^n$ contains $x = 0$.

Let $V : D \rightarrow \mathbb{R}$ be continuously differentiable function w.r.t
$$V(0) = 0 \quad \text{and} \quad V(x) > 0 \quad \text{for} \ x \in D \cap B_{\delta/2}$$
$$V(x) \leq 0 \quad \text{for} \ x \in D$$

Then $x = 0$ is stable.

Moreover, if $\dot{V}(x) < 0$ for $x \in \partial B_{\delta/2}$

Then $x = 0$ is asymptotically stable.
Given \( \varepsilon > 0 \), choose \( \lambda \in (0, \varepsilon) \), s.t.

\[
B_\lambda = \{ x \in \mathbb{R}^n \mid \| x \| \leq \lambda \}
\]

and \( \lambda \) is a positive \( \lambda \).

Let \( \alpha = \min \{ \| x \| \mid V(x) \leq \beta \} \).

Take \( \beta \in (0, \alpha) \), and let

\[
\mathcal{N}_\beta \supset \{ x \in B_\lambda \mid V(x) \leq \beta \}
\]

\( \mathcal{N}_\beta \) is a subset of \( B_\lambda \). Also \( \mathcal{N}_\beta \) is an invariant set:

\[
V(x(t)) \leq 0 \Rightarrow V(x(t)) \leq V(x(0)) \leq \beta, \forall t \geq 0
\]

\( V(x) \) is continuous and \( V(0) = 0 \), \( \exists \delta > 0 \), s.t.

\[
\| x \| < \delta \Rightarrow V(x) < \beta
\]

\( B_\delta \subset \mathcal{N}_\beta \subset B_\lambda \)

and \( x(0) \in B_\delta \Rightarrow x(t) \in \mathcal{N}_\beta \Rightarrow x(t) \in B_\lambda \)

\[
\forall \| x(t) \| < \delta, \forall t \geq 0
\]

To show asymptotic stability:

Show \( x(t) \to 0 \) as \( t \to \infty \)

or for every \( \alpha > 0 \), \( \exists T > 0 \), s.t. \( \| x(t) \| < \alpha, \forall t > T \)

we can show (as above) that for every \( \varepsilon > 0 \), we can choose \( \beta > 0 \), s.t. \( \mathcal{N}_\beta \subset B_\lambda \).

Show that \( V(x(t)) \to 0 \) as \( t \to \infty \)

\( V(x(t)) \) is monotonically decreasing and bounded below by \( 0 \)

\( V(x(t)) \to C \neq 0 \) as \( t \to \infty \)

we need to show that \( C = 0 \).

**Proof by Contradiction:** Assume that \( C > 0 \). By continuity of \( V(x) \), \( \exists T > 0 \), s.t. \( \mathcal{N}_T \subset B_\lambda \) for all \( T > 0 \). Let \( -V = \max_{x \in B_\lambda} \{ \| x \| \leq \lambda \} \)

which exists on the compact set for a continuous function. \( -V < 0 \)

\[
- V(x(t)) = V(x(0)) + \int_0^t V(x(s)) \, ds \leq V(x(0)) - T
\]

\( \Rightarrow \) as \( t \to \infty \), \( V(x(t)) \) becomes negative. \( \equiv \) contradiction

\( \Rightarrow C = 0 \).
Example 1. \( \dot{\theta} + \theta + \sin \theta = 0 \), equilibrium at \( \theta = 0, \dot{\theta} = 0 \)

\[
V(x) = (1 - \cos \theta) + \frac{\dot{\theta}^2}{2}
\]

\[
V(x) = \dot{\theta} \sin \theta + \dot{\theta} \dot{\theta} = -\dot{\theta}^2 \leq 0
\]

\( \therefore \dot{\theta} = 0 \) is stable

Example 2.

\[
\begin{align*}
\dot{x}_1 &= x_1 (x_1^2 + x_2^2 - 1) - x_2 \\
\dot{x}_2 &= x_1 + x_2 (x_1^2 + x_2^2 - 1)
\end{align*}
\]

\[
V(x_1, x_2) = x_1^2 + x_2^2
\]

\[
\dot{V} = 2 (x_1^2 + x_2^2) (x_1^2 + x_2^2 - 1)
\]

\( \therefore \dot{V} < 0 \) \& \( x_1, x_2 \in (x_1^2 + x_2^2 < 1) \)

\( \therefore \) locally asymptotically stable equilibrium

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**Lyapunov Theorem for Global Stability (Asymptotic)**

- Let \( x = 0 \) be the equilibrium point for \( \dot{x} = f(x) \). Let \( V: \mathbb{R}^n \to \mathbb{R} \) be a continuously differentiable function, s.t.
  \[
  V(0) > 0 \quad \text{and} \quad V(x) > 0, \forall x \neq 0
  \]
  \[
  |x| \to \infty \Rightarrow V(x) \to \infty \quad \text{(radial unboundedness)}
  \]
  \[
  \dot{V} < 0, \forall x \neq 0
  \]

Then \( x = 0 \) is globally asymptotically stable.

**Proof:** Given any \( p \in \mathbb{R}^n \), let \( C = V(p) \).

- Of radial unboundedness, \& \( \forall C > 0, \exists \epsilon > 0 \) s.t. \( V(x) > C \) whenever \( |x| > \epsilon \).
  \[
  V(x) > C \quad \text{whenever} \quad |x| > \epsilon \quad \Rightarrow \quad \text{bounded}.
  \]

Check of proof as local asymptotic stability.

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[Diagram of a graph with a curve labeled \( V(x) \) and arrows indicating the flow of \( \dot{x} = f(x) \) with \( x(0) \) showing \( \dot{V} < 0 \), and a point \( x(\infty) \) approaching \( x(0) \) as \( t \to \infty \).]