SYSTEM DYNAMICS AND FEEDBACK CONTROL
DESIGN PROBLEM FORMULATIONS FOR REAL TIME
RAMP METERING

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This paper formulates the ramp metering control problem as a real-time feedback control
problem. The development is shown by modeling freeway sections, and hence the results are
immediately applicable to isolated ramp metering problems. The ideas presented in this paper
can be utilized to develop coordinated ramp metering feedback control techniques. Three
different forms of the formulation for are presented: (1) distributed parameter system form
derived from the conservation law; (2) space discretized continuous lumped parameter form;
(3) space and time discretized lumped parameter form. These formulations can be considered
as the starting points for development of feedback control laws for the different control problems
stated in this paper. This paper presents the feedback control problems and a new nonlinear
control design for an isolated ramp metering problem. Simulations are performed for the isolated
case using the new nonlinear feedback control law which show very encouraging results.

1. Introduction

Ramp metering is a way to improve traffic flow by regulating the ramp inflow to a freeway. By
effectively controlling the ramp flow, the traffic density on the mainline freeway can be kept below
critical level to provide high throughput that is congestion free. For this type of operation, many factors
have to be considered such as:

• The inflow at the mainline
• The queue holding capacity of the ramp
• Availability of sensors
• The arterial system connected to the ramp system
1.1. Literature Review

The ramp flow problem has been studied for more than thirty years. Some of the early work is documented in references [1-4]. This work was related to merge control and ramp metering control design based on demand-capacity relationships. Some early deployment studies were also performed at various sites such as Chicago and Houston. Reference [5] provides a current overall overview of ramp metering. References [6-8] show the work that used optimization techniques for solving optimal ramp control problems. Some current evaluation studies and simulation based evaluation methods are described in [9-10]. Some researchers have designed feedback control laws for ramp metering [11-14]. These laws are designed after performing linearization of the dynamics about the nominal equilibrium state. Recently, ramp meters have been deployed in many places internationally, such as in U.S.A [15], France [16], Italy [17], Germany [18], New Zealand [19], U.K. [20], and Netherlands [21].

Feedback control is a very powerful technique for ramp control since it is traffic responsive and has the least computation cost, hence is a real-time control strategy. However, until now, mainly linear control design has been studied, which is powerful but provides only local results. For global results, nonlinear techniques become necessary. This paper is written to carefully study the ramp control problem based on different system dynamics models, and it proposes some nonlinear controllers for some isolated ramp control cases.

1.2. Problem Description

Ramp metering can help in providing a smooth flow of traffic on urban freeways. Moreover, it can also help in alleviating congestion on the freeways. The design of ramp metering entails measuring some traffic variables on the freeway and adjusting the ramp metering rate to provide the smooth flow. This structure of performing measurements using sensors and in real time adjusting the ramp metering rates renders the problem as that of a closed loop feedback control problem.

Ramp metering attempts to keep mainline volumes below capacity by controlling the number of vehicles entering the freeway. Under ideal conditions, the wait on the entrance ramp would be compensated for by increased speeds once on the freeway. Ramp meters can increase freeway speeds while providing increased safety in merging and reducing rear-end collisions on the ramps themselves. The topology of a ramp metering system is shown in Figure 1.

Feedback control for ramp metering can be an effective solution for alleviating traffic congestion. The designer of the controller needs to address issues such as controllability and observability of the traffic system, actuation and sensing, robustness, and stability of the closed loop system shown in Figure 1. The actuation of this system can be achieved by the light signal which indicates whether the vehicles can go into the freeway or not. State variables such as the traffic density, average traffic speed, etc. can be sensed using various types of traffic sensors such as inductive loops, traffic cameras, transponders, etc.

2. System Modeling

The first step in the design of feedback controllers for ramp metering is to model the system dynamics appropriately. Macroscopic model of the traffic can effectively be used in this context. From the macroscopic perspective, the traffic flow is considered analogous to a fluid flow, which is a distributed parameter system represented by partial differential equations. Mass conservation model of a freeway, characterized by \( x \in [0, L] \), which is the position on the freeway, is given by:

\[
\frac{\partial}{\partial t} \rho(x, t) = -\frac{\partial}{\partial x} q(x, t)
\]
where $\rho(x,t)$ is the density of the traffic as a function of $x$, and time $t$, and $q(x,t)$ is the flow at given $x$, and $t$. The flow $q(x,t)$ is a function of $\rho(x,t)$, and the velocity $v(x,t)$, as shown below:

$$q(x,t) = \rho(x,t)v(x,t)$$

(2)

This model of a highway section is shown in Figure 2.

There are various static and dynamic models which have been used to represent the relationship between $v(x,t)$ and $\rho(x,t)$. One of the most simple models is the one proposed by Greenshield [22], which hypothesizes a linear relationship between the two variables.

$$v = v_f (1 - \frac{\rho}{\rho_{max}})$$

(3)

where $v_f$ is the free flow speed, and $\rho_{max}$ is the jam density.

We can design the control law directly using the partial differential equation (pde) model of the system and hence not introduce discretization errors in the model. On the otherhand, the sensor measurements are usually made at discrete points, and also in general control design in pde setting is usually more difficult. In anycase, it is important to analyze control laws designed using different paradigms, so that they can be compared for their effectiveness.
3. Feedback Control for the Traffic as a Distributed Parameter System

The system represented by equations (1-3) is an infinite dimensional representation of the traffic, since it has infinite state variables. There are two ways to mathematically design a ramp metering control feedback controller for such a system. One way is to work in the infinite dimensional domain, and design a controller, which then can be discretized. Another way is to space discretize (Partial Differential Equation) PDE (1) to obtain an (Ordinary Differential Equation) ODE representation of the system. The ODE representation is a standard representation for most of the results available in feedback control theory, and therefore, using that representation many techniques from control theory can be utilized.

The modeling of traffic in the PDE domain provides a reasonably accurate model of the traffic system, especially since phenomena such as shock waves are effectively represented. Hence, it would be highly desirable to design a feedback controller directly utilizing this model. However, it is not trivial to design feedback control laws using this modeling scheme. Fortunately, transforming the traffic model into a more convenient form of Burgers’ equation reduces the complexity of the control design. Burgers’ equation is just a way of representing the same hydrodynamic traffic flow model in a different form utilizing the diffusion behavior of the traffic. Hence this model is not different in physical interpretation than the classical models which are being widely used in the traffic modeling community.

In the PDE context, researchers have used Burgers’ equation to model the traffic flow [23, 24]. There has been some research work conducted on the problem of computing feedback laws for Burgers’ equation, which is a nonlinear PDE [25, 26]. Burgers’ equation applied to traffic flow is given by:

$$\frac{\partial}{\partial t} \rho(x, t) + \rho(x, t) \frac{\partial}{\partial x} \rho(x, t) = \varepsilon \frac{\partial^2}{\partial x^2} \rho(x, t)$$  \hspace{1cm} (4)

and was introduced by Burgers [27, 28, 29] as a simple model for turbulence, where $\varepsilon$ is a viscosity coefficient. The following is borrowed from [23] to show how the traffic problem can be modeled as Burgers’ equation.

In order to account for the fact that drivers look ahead and modify their speeds accordingly, Equation (3) can be replaced by:

$$v_e = v_f (1 - \frac{\rho}{\rho_{\text{max}}}) - D(\frac{\partial \rho}{\partial x})/\rho$$  \hspace{1cm} (5)

Using (5) and the fact $q = \rho v_e$, relationship (2) now can be replaced by:

$$q(x,t) = \rho(x,t)v(x,t) - D(\frac{\partial \rho}{\partial x})$$  \hspace{1cm} (6)
where $D$ is a diffusion coefficient [19] given by:

$$D = \tau v_r^2$$  \quad (7)

where $v_r$ is a random velocity, and $\tau$ is a parameter. Diffusion is a useful concept mentioned by many researchers as an extension to the existing traffic flow models to improve their realism [23, 24, 30]. Diffusion term represents “the diffusion effect” due to the fact that each driver’s gaze is concentrated on the road in front of him/her, so that he/she adjusts his/her speed according to the concentration ahead. This adjustment creates a dependence of flow on concentration gradient which leads to an effective diffusion. This models the gradual rather than instantaneous reduction of speed by the drivers in response to shock waves. The diffusion term $D$ has the units of velocity multiplied by those of length, such as mile$^2$/hour. Some researchers have also used the Burgers’ equation to represent traffic system [23, 24]. Combining Equations (1) and (6) gives:

$$\frac{\partial}{\partial t} \rho(x,t) + v_f \frac{\partial}{\partial x} \rho(x,t) - 2 \frac{\rho}{\rho_{\text{max}}} v_r \frac{\partial}{\partial x} \rho(x,t) - D \frac{\partial^2}{\partial x^2} \rho(x,t) = 0$$  \quad (8)

If we introduce a moving reference frame:

$$\xi(x,t) = -x + v_f t$$  \quad (9)

and non-dimensionalize $\rho(x,t)$ by dividing it by $\rho_{\text{max}} / 2$, and $t$ by $t_0$, equation (7) gets transformed to:

$$\frac{\partial}{\partial t} \rho(\xi, t) + \rho \frac{\partial}{\partial \xi} \rho(\xi, t) - \frac{1}{Re} \frac{\partial^2}{\partial \xi^2} \rho(\xi, t) = 0$$  \quad (10)

Here, $Re$ is a dimensionless constant, and is analogous to the Reynolds number in fluid dynamics. $Re$ is given by:

$$Re = \left( \frac{v_f}{v_r} \right)^2 \frac{t_0}{\tau}$$  \quad (11)

Equation (10) shows the Burgers’ equation formulation of the traffic flow problem. Some researchers have also worked on the conservation law given by Equation (12):

$$\frac{\partial}{\partial t} \rho(x,t) + \rho(x,t) \frac{\partial}{\partial x} \rho(x,t) = \varepsilon \frac{\partial^2}{\partial x^2} \rho(x,t)$$  \quad (12)

with a solution obtained by taking the following limit:

$$\rho(x,t) = \lim_{\varepsilon \rightarrow 0} \rho^f(x,t)$$  \quad (13)
where $\rho^e(x, t)$ satisfies (4) [31-36]. Using this form reduces the Burgers’ equation formulation into the classical traffic model with no diffusion. So, even in the PDE domain, we can try to work on the same model as the classical traffic models.

Work on the feedback control of Burgers’ equation has been performed [26, 27, 37] by some researchers in the past. Curtain [37] showed using Kielhofer’s stability results for semi-linear evolution equations [38], that there exists a stabilizing feedback law which can be obtained from the linearized equation, when the domain of the output operator is a certain subspace of $L^2$ which contains the Sobolev space $H^1_0$, where for a given domain $\Omega$ with boundary $\partial\Omega$, $L^2(\Omega)$ is the space of all measurable functions $f$ such that $\int_\Omega |f(x)|^2 \, dx < \infty$, and $H^1_0(\Omega)$ is the set of all functions $f$ in $L^2(\Omega)$ such that the derivatives $f'$ (or $\nabla f$) are also in $L^2(\Omega)$ and $f|_{\partial\Omega} = 0$, implying that $f=0$ is on the boundary. Burns and Kang [25] show the design of Linear Quadratic Regulator (LQR) optimal controller for the linearized equation [26]. They also studied a boundary control problem [26] which is relevant for the traffic control problem, since the control split factor enters the dynamics as a boundary injection.

The local ramp metering control problem in the distributed parameter domain is essentially a boundary injection feedback control problem, where the control input enters the traffic system dynamics through the boundary condition. One solution to that type of problem relies on converting it into a control problem linear in control and applying LQ methods.

3.1. Ramp Control Formulation

Consider the traffic flow in a freeway section with a ramp as shown in Figure 3. We consider the spatial variable $x$ to be zero at the location where the entrance ramp is located.

We can write the Burgers’ equation for the freeway and include ramp flow as a boundary input. The following is the conservation equation (from which the Burgers’ equation is derived) for the freeway:

$$\frac{\partial}{\partial t} \rho(x, t) + \frac{\partial}{\partial x} q(x, t) = 0, \quad 0 \leq x \leq L,$$

(14)

with the following boundary conditions at $x=0$.

$$q(0, t) = q_{in}(0, t) + r(t).$$

(15)

![Fig. 3 Traffic flow with a ramp.](image-url)
The output bounded equation for this traffic system, based on the sensors used is:

\[ y_i(t) = \varphi_i(q(x,t)), \quad i = 1, 2, \ldots, m, \quad (16) \]

where the bounded function \( \varphi_i(\cdot) \) is given by:

\[ \varphi_m(q(x,t)) = \frac{1}{2\delta} \int_{x_m-\varepsilon}^{x_m+\varepsilon} q(x,t) \, dx. \quad (17) \]

The control aim is to ensure that \( \rho(x,t) < \rho_m(x) \quad \forall x \in (0,L), t \in \mathbb{R}^+ \), where \( \rho_m \) is the jam density. This can be achieved by having a critical density value close to the maximum flow density as the target density for the controller. The nonlinear optimal control problem can be stated as: find \( r_o(t) \), the optimal \( r(t) \), which minimizes

\[ J(r) = \int_0^{t_f} \left[ \left( r - r_{max} \right)^2(t) + w \cdot \int_0^L \left[ \rho(x,t) - \rho_{cr}(x,t) \right]^2 \, dx \right] \, dt, \quad (18) \]

\( t_f \) is the final time, \( \rho_{cr} \) is the design critical density (which is the density corresponding to the maximum traffic), \( r_{max} \) is the maximum possible ramp flow into the mainline, and \( w \) the relative weight. Note that a feedback solution is needed for the problem, and not an open loop optimal control solution. Hence, we can either decide the structure of the feedback control such as a PID control with constant gains, and solve numerically for the optimal values of the gains, or we can state the control objective for a standard feedback control problem, such as steady state asymptotic stability

\[ \int_0^L \left[ \rho(x,t) - \rho_{cr}(x,t) \right]^2 \, dx \to 0 \quad (19) \]

and some transient behavior characteristics like some specified settling time, percent overshoot, etc. We could also linearize the nonlinear system, and formulate the LQR problem solution of which is a state feedback control problem, or try some extensions of techniques developed for control of nonlinear ODE’s to nonlinear PDE’s.

4. Discretized System Dynamics

Many researchers have studied and designed optimal open loop controllers utilizing space and time discretized models of traffic flow. Some researchers have also designed feedback control laws using similar models. The reason for the popularity of these models is that there are many techniques available to deal with discretized systems. The same is also true for feedback control, and hence, in order to utilize the various linear and nonlinear [39-43] control techniques available for lumped parameter systems, the distributed parameter model is space discretized [39]. For this the freeway is subdivided into several sections as shown in Figure 4.
Space discretization is performed by dividing the considered highway links into segments. In general, the length of each segment is taken to be between .5 mile and 1 mile. This is an approximation that is quite realistic since the traditional sensors like loop detectors along a freeway, are generally installed at least 1 mile apart. Although a smaller step size for space discretization will undoubtedly improve the accuracy of the simulation, in reality it is not possible to measure speed and flow variables at smaller intervals due to limited availability of sensors along freeways. Thus, 1 mile segment length for space discretization appears to be a realistic assumption. On the other hand, the time discretization can be done using very small time steps since traffic data can be downloaded from sensors practically at every second. With the use of more sensors such as CCTVs we can obtain information at higher resolution to have smaller space discretization steps.

The space discretized form of Equation (1) produces the following continuous ODEs for the $n$ sections of the highway.

$$\frac{d}{dt} \rho_i = \frac{1}{\delta_i} [q_i(t) - q_{i+1}(t) + r_i(t) - s_i(t)], \quad i = 1, 2, ..., n. \quad (20)$$

Here, $r_i(t)$ and $s_i(t)$ terms indicate the on-ramp and off-ramp flows. Equation (20) combined with (2), (3) (or (4), (5) depending on the decision to include the diffusion term), and the output equations (19) give the mathematical model for a highway, which can be represented in a standard nonlinear state space form for control design purposes.

$$y_j = g_j(\rho_1, \rho_2, ..., \rho_n), \quad j = 1, 2, ..., p. \quad (21)$$

The standard state space form is:

$$\frac{d}{dt} x(t) = f[x(t), u(t)],$$
$$y(t) = g[x(t), u(t)],$$
$$x(0) = x_0, \quad (22)$$

where $x = [\rho_1, \rho_2, ..., \rho_n]^{T}$ and $u(t) = r(t)$.

There are various other proposed models, which are more detailed in the description of the system dynamics. The phenomenon of shock waves, which is very well represented in the PDE representation of the system, is modeled by expressing the traffic flow between two contiguous sections of the highway as the weighted sum of the traffic flows in those two sections which correspond to the densities in those two sections [44, 45]. A dynamic relationship, instead of a static one like (3), has also been proposed by [46] and used successfully.
The model thus obtained can also be time discretized to transform the continuous time model into a discrete time mode. A comprehensive model, which incorporates shock waves, as well as represents the dynamic nature of mean speed propagation, is shown in [39] and is reproduced here for completion. The difference equations:

\[
\rho_j(k+1) = \rho_j(k) + \frac{T}{\delta_j} [q_{j-1}(k) - q_j(k) + r_j(k) - s_j(k)],
\]

\[
v_j(k+1) = v_j(k) + \frac{T}{\tau} [v_e(\rho_j(k)) - v_j(k)] + \frac{T}{\delta_j} v_j(k)[v_{j-1}(k) - v_j(k)] - \frac{\sqrt{T}}{\tau \delta_j} \frac{\rho_j(k+1) - \rho_j(k)}{\rho_j(k) + \delta_j}
\]  

(23)

with the relationships:

\[
q_j(k) = \alpha \rho_j(k) v_j(k) + (1 - \alpha) \rho_{j+1}(k) v_{j+1}(k), \quad 0 \leq \alpha \leq 1,
\]

\[
v_e(\rho) = v_f [1 - \left(\frac{\rho}{\rho_{\text{max}}}\right)^m],
\]

(24)

output measurements of traffic flows q and time mean speeds y, shown as:

\[
y_j(k) = \gamma v_j(k) + (1 - \gamma) v_{j+1}(k), \quad 0 \leq \gamma \leq 1,
\]

(25)

and the boundary conditions:

\[
v_0(k) = y_0(k),
\]

\[
\rho_{n+1}(k) = q_n(k)/y_n(k),
\]

(26)

gives the discrete system dynamics, which can be represented in the standard nonlinear discrete time form:

\[
x(k+1) = f(x(k), u(k)),
\]

\[
y(k) = g(x(k), u(k)),
\]

\[
x(0) = x_0,
\]

(27)

where control u(k) is the vector OD ramp input flows.

If the control actuation is discrete, such as the ones implemented by microprocessors and computers, feedback control laws can be designed based on the discrete model (27), or can be designed using (22) after which the controller can be discretized.

5. Feedback Ramp Control for the Traffic as a Lumped Parameter System

In the discretized traffic flow model, the freeway is divided into sections with aggregate traffic densities. Sensors are used to measure variables such as densities, traffic flow and traffic average speeds in these sections, which can be used by the feedback controller to give appropriate commands to the ramp signal.

There are essentially two ways to design controllers for such nonlinear traffic systems of the form (22) and (27). One way is to design the controller by linearizing the nonlinear dynamics of the system about its equilibrium, or a trajectory; the other way is to design directly for the nonlinear system. The first way is easier, since immense literature is available describing the various design techniques, especially for LTI systems, but the results are valid only where the linearization is applicable. On the other hand,
design of controllers directly for the nonlinear system is much more difficult, but the results are usually global. Some of the linear control techniques are LQR (Linear Quadratic Regulator), LQG (Linear Quadratic Gaussian), PID (Proportional Integral Derivative), $H_\infty$, preview control, etc., and some of the nonlinear ones are describing functions design, feedback linearization, sliding mode control, nonlinear $H_\infty$, etc. Qualitative methods such as fuzzy control and expert systems can also be utilized. Qualitative methods are usually easy to design but difficult to tune and analyze.

5.1. Ramp Metering Formulation

Let the freeway be divided into $n$ sections. For simplicity, we are considering static velocity relationships, and ignoring the effect of downstream flow for the model:

$$\frac{d}{dt}\rho_i = \frac{1}{\delta_i} [q_{i-1}(t) - q_i(t) + \chi(i) \cdot r_i(t) - \zeta(i) \cdot s_i(t)], \quad i = 1, \ldots, n$$

(28)

with relationships (2) and (3). The functions $\chi(i)$ and $\zeta(i)$ are presence functions for on and off ramps, i.e. if there is an on ramp in section I then $\chi(i)=1$, otherwise it is zero. Let us form a $p$ order vector ($p$ being the number of controllable on-ramps on the freeway) formed by putting all the $r_i(t)$ s for which $\chi(i)=1$ (assuming all the ramps for which this is true are controllable, otherwise the control vector would just be a subset of this vector). Let us call this input vector $\mathbf{r}$. The control problem can be stated as: find $\mathbf{r}_o(t)$, the optimal $\mathbf{r}(t)$, which minimizes

$$J(\mathbf{r}) = \int_0^{t_f} [(\mathbf{r} - \mathbf{r}_{\text{max}})^T (\mathbf{r} - \mathbf{r}_{\text{max}}) + w \cdot \sum_{i=1}^{n} (\rho_i - \rho_{cr i})^2] \, dt$$

(29)

where $t_f$ is the final time, $\rho_{cr i}$ is the design critical density for section $i$, and $w$ a relative weight. Note that, just like for the distributed parameter system case, a feedback solution is needed for the problem, and not an open loop optimal control. Hence, here also, we can either decide the structure of the feedback control, such as a PID control with constant gains, and solve numerically for the optimal values of the gains, or we can state the control objective for a standard feedback control problem, such as steady state asymptotic stability

$$\lim_{t \to \infty} \sum_{i=1}^{n} (\rho_i - \rho_{cr i})^2 \to 0$$

(30)

and some transient behavior characteristics like some specified settling time, percent overshoot, etc. Note that the design has also to satisfy the constraint that the section densities should remain below section jam densities at all times.

The discrete time versions of (28-29) are given below:

$$\rho_i(k+1) = \rho_i(k) + \frac{T}{\delta_i} [q_{i-1}(k) - q_i(k) + \chi(i) \cdot r_i(k) - \zeta(i) \cdot s_i(k)], \quad i = 1, \ldots, n$$

(31)
\[ J(r) = \sum_{k=1}^{k_f} [(r(k) - r_{\text{max}})^T(r(k) - r_{\text{max}}) + w \cdot \sum_{i=1}^{n} (\rho_i(k) - \rho_{\text{cr}_i}(k))^2], \] (32)

\[ \lim_{k_f \to \infty} [\sum_{i=1}^{n} (\rho_i(k) - \rho_{\text{cr}_i}(k))^2] \to 0. \] (33)

Note that in order to keep the ramp rates as high as possible, we keep the weight on the difference between the ramp rate and the maximum rate in the objective function. If sensors are located on the ramps to measure the queue lengths, then we can also add queue length dynamics to the system and then we would include the queue length in the objective function for minimization.

Based on the standard forms described in this section, the appropriate feedback control laws can be designed. The exact methodology of the design will be studied in subsequent papers, but the next section elucidates an example to show how the feedback control could be used.

6. Sample Problems

We present two control laws for isolated ramp metering control: one using space discretized dynamics and the other one using space and time discretized dynamics. This is an important class of ramp metering problems and has been studied by many researchers [1, 11, 13, 14]. The results we present in this section for this problem are new, since we are analytically deriving the control law by performing the design in the nonlinear setting and providing the control law.

6.1. Space Discretized Case

In order to illustrate the ideas discussed above, we have designed a feedback control law for a space discretized system. The control is based on feedback linearization technique for nonlinear systems. The isolated ramp metering problem area is shown in Figure 5 below.

The dynamic equation for this problem is given by:

\[ \frac{d}{dt} \rho = \frac{1}{L} [q_{\text{in}}(t) - q_{\text{out}}(t) + r(t)] \] (34)

Because of difficulties in measuring traffic density, we use vehicle occupancy as the state variable. The relationship between traffic density and vehicle occupancy, \( o(t) \), in Equation (35), \( k \) is a constant, \( \mu \) is the number of lanes on the freeway section, and \( \lambda \) is the mean effective vehicle length [11]. The dynamics of the system can now be represented as:

\[ \rho(t) = \alpha o(t), \text{ where } \alpha = k\mu/\lambda \] (35)

Fig. 5  Isolated ramp metering problem example.
In Equation (35), \( k \) is a constant, \( \mu \) is the number of lanes on the freeway section, and \( \lambda \) is the mean effective vehicle length [11]. The dynamics of the system can now be represented as:

\[
\dot{\rho}(t) = [q_{in}(t) - q_{out}(t) + r(t)] / (\alpha L)
\]

(36)

The control law for this system using feedback linearization is:

\[
r(t) = -K[\rho(t) - \rho_{cr}] + q_{out}(t) - q_{in}(t)
\]

(37)

Here, \( K \) is the control gain and \( \rho_{cr} \) is the critical value of occupancy corresponding to the critical value of the traffic density.

**Theorem 1**: The control law (37) when applied to the system (36) guarantees that \( \lim_{t \to \infty} (\rho - \rho_{cr}) \to 0 \), which is the objective of the controller. In fact it guarantees that the rate of convergence of \( \rho - \rho_{cr} \) is exponential at a rate dictated by the control gain \( K \).

**Proof**: Substituting \( r(t) \) from (37) to (36) gives:

\[
\dot{\rho}(t) = -K[\rho(t) - \rho_{cr}] / (\alpha L)
\]

(38)

Now if we define

\[
o = \rho - \rho_{cr}
\]

(39)

and note that \( \rho_{cr} = 0 \), because \( \rho_{cr} \) is a constant, then we obtain

\[
o(t) = -ko(t)
\]

(40)

Solving (40) we obtain the time history of the occupancy error as a function of time.

\[
o(t) = o(0)e^{-kt}
\]

(41)

Equation (41) shows that \( \lim_{t \to \infty} (\rho - \rho_{cr}) \to 0 \).

**6.1.1 Comparative Analysis**

Comparison with Wattleworth Model [1]: The control law Wattleworth proposed is

\[
r(t) = q_{out}(t) - q_{in}(t)
\]

(42)

This law is derived from the steady state behavior of (36) which is obtained by equating the left hand-side to zero to yield:

\[
q_{in}(t) - q_{out}(t) + r(t) = 0
\]

(43)

This relationship is shown in Figure 6 (Static Model).
On the other hand, the control law proposed here is designed to control the transients as well as steady state behavior and uses the model shown in Figure 6 (Dynamic Model). Note that when $\omega = \omega_{cr}$ the two control laws (37) and (39) become the same, implying that the steady state control law is the same as (39).

Comparison with ALINEA Model: The ALINEA model uses the following control law [11]:

$$r(i) = r(i-1) + K[\omega(i) - \omega_{cr}]$$  \hspace{1cm} (44)

The differences between ALINEA (44) and (37) are the following:

1. ALINEA law is a discrete controller as compared to the proposed continuous controller.
2. As is obvious from comparison of (44) and (37) ALINEA does not perform nonlinear canellation.
3. ALINEA was designed using linearization of the system dynamics [11], and hence is valid for local region about the equilibrium, as compared to the proposed law which cancels the nonlinearities explicitly, and hence is valid globally.

6.2. Sample Problem for Space and Time Discretized Case

The dynamic equation for the space and time discretized form is:

$$\rho(k+1) = \rho(k) + \frac{T}{L} [q_{in}(k) - q_{out}(k) + r(k)]$$  \hspace{1cm} (45)

Applying the occupancy relationship as before, we obtain:

$$\omega(k+1) = \omega(k) + \frac{T[q_{in}(k) - q_{out}(k) + r(k)]}{(\alpha L)}$$  \hspace{1cm} (46)
Fig. 7 Isolated ramp control simulation with initial density of 40 vehicles/mile.

We can design the control law following discrete time feedback linearization technique, resulting in:

\[ r(k) = K[\omega(k) - \omega_{cr}] \cdot (\alpha L / T) + [q_{out}(k) - q_{in}(k)] \]  

(47)

This control law guarantees that \(\lim_{k \to \infty} (\rho - \rho_{cr})^2 \to 0\), which is the objective of the controller. In fact it guarantees that the rate of convergence of \(\omega - \omega_{cr}\) is geometric at a rate dictated by the control gain \(K\). The control law gains of (37) and (47) can be chosen to provide a reasonable rate of error decay. For practical purposes, gain values can be fine-tuned in the field by performing data collection in the evaluation phase for various gain values. Notice that all these control laws including the Wattleworth, ALINEA, and the proposed control laws do not explicitly take the ramp queues into account. Active research in designing feedback control laws incorporating the queue lengths also is being actively persued currently by the author.
6.2.1. Comparative Analysis

For comparison with Wattleworth’s control law, the argument of section 6.1.1 applies, including the fact that the steady state control law (47) becomes the same as (42). Compared to control laws proposed in reference [11], this law has a nonlinear feedforward term to cancel the nonlinearities, and hence is a global control law, as compared to the law in [11] which is valid for only small perturbations about the steady state equilibrium. The law in [14] is a neural network based control law and hence, in general does not provide any analytic guarantees of performance.

6.2.2. Simulation Results

We carried out a number of simulations. We used Greenshield’s flow-density relationship with $v_f = 60$ miles/hour, and $\rho_m = 120$ vehicles per mile [14]. This produces a critical density $\rho_c = 60$ vehicles/mile with a maximum flow of 1800 vehicles/hour. We choose the target signal to be $\rho_c = 55$ vehicles per mile.

In the first simulation, we used 40 vehicles/mile as the initial density which is in the uncongested region of the flow-density relationship. The inflow to the main line was kept at 1500 vehicles/hour. The controller (Equation 47) is effectively able to stabilize the freeway flow as seen in Figure 7.

In the second simulation, we used 65 vehicles/mile as the initial density which is in the congested region of the flow-density relationship. The inflow to the main line was kept at 1500 vehicles/hour. The controller (Equation 47) in this case also is effectively able to stabilize the freeway flow as seen in Figure 8.
In the third simulation, we used 40 vehicles/mile as the initial density. The inflow to the main line was kept random between the values of 1500 and 1700 vehicles/hour. The controller in this case also is effectively able to stabilize the flow as seen in Figure 9.

7. Conclusion

In this paper, we formulated feedback control problems in the distributed parameter setting as well as in continuous-time and discrete-time lumped parameter setting for ramp metering. Brief discussions on applicable real-time feedback control methods were provided. This paper should provide incentive to further the development of feedback control design and application for ramp metering control problems in real-time. Feedback control models, by making use of the sensor data and minimizing the need for expensive computational requirements of classical optimization techniques, provide a viable alternative in the context of ITS and at the same time provide a robust solution in the presence of disturbances and uncertainties.

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9. References


