SOLUTION TO THE USER EQUILIBRIUM DYNAMIC TRAFFIC ROUTING PROBLEM USING FEEDBACK LINEARIZATION

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Abstract—In this paper, the Dynamic Traffic Routing problem is defined as the real-time point diversion of traffic during non-recurrent congestion. This dynamic traffic routing problem is then formulated as a feedback control problem that determines the time-dependent split parameters at the diversion point for routing the incoming traffic flow onto the alternate routes in order to achieve a user-equilibrium traffic pattern. Feedback linearization technique is used to solve this specific user-equilibrium formulation of the Dynamic Traffic Routing problem. The control input is the traffic split factor at the diversion point. By transforming the dynamics of the system into canonical form, a control law is obtained which cancels the nonlinearities of the system. Simulation results show that the performance of this controller on a test network is quite promising.

Keywords: control, closed loop, traffic diversion

1. NOTATION

Traffic variables

\[ q_i \] traffic volume entering link \( i \)

\[ q_{i,j} \] traffic volume entering link \( j \) of route \( i \)

\[ \rho_i \] traffic density in link \( i \)

\[ \rho_{i,j} \] traffic density in link \( j \) of route \( i \)

\[ v_i \] average traffic speed in link \( i \)

\[ r_i \] ramp traffic flow entering link \( i \)

\[ s_i \] ramp traffic flow exiting link \( i \)

\[ \beta \] traffic split factor at a node

\[ U \] input flow at a node

\[ x \] state vector

\[ y \] measurement vector

\[ u \] input vector

\[ e \] error vector

\[ J \] objective function

\[ \chi(\ldots) \] travel time function

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2. INTRODUCTION

Real-time control of traffic diversion during non-recurrent congestion continues to be a challenging topic. With the advent of intelligent transportation systems (ITS), the need for real-time models and algorithms that will control the diversion becomes especially evident. Several researchers tried to solve this on-line control problem by adopting different approaches. Among the models that have been developed for determining diversion routes and diverting the traffic onto these routes in the context of advanced traffic management systems (ATMS)/advanced traveller information systems (ATIS) are expert systems, feedback control, and mathematical programming models.

Expert system-based strategies have been used for diversion (Gupta et al., 1992; Ketselidou, 1993). Gartner and Reiss (1987) developed a multilevel control system incorporating local-level and corridor-level controls. This framework recognizes and emphasizes the completion of the feedback loop between the system outputs and the control inputs. Perhaps the most popular way of solving the dynamic traffic assignment (DTA)/dynamic traffic routing (DTR) problem is through the use of traditional optimization-based approaches that attempt to optimize the objective functions for the nominal model over the ‘planning horizon’ (Peeta and Mahmassani, 1995). Papageorgiou (1990) and Papageorgiou and Messmer (1991) have designed some bang-bang controllers, controllers based on the linear quadratic regulator (LQR) principle, and controllers based on the rolling horizon technique. Kachroo and Özbay (1996) designed a fuzzy feedback control law for point diversion.

The DTR feedback controller discussed in this paper will be used as part of an overall incident management framework in order to alleviate congestion caused by incidents (Hobeika et al., 1993; Kachroo et al., 1997b). Real-time route guidance differs from the diversion concept used in the planning concept in that it is not a long-term policy decision concerning route choice, but rather it is a reaction to an immediate situation facing the motorist (Hall, 1974). The static and quasi-dynamic techniques for traffic assignment (Reiss et al., 1991) are useful for achieving pre-set system optimal or user-equilibrium conditions but cannot be used for the feedback control problem.

For real-time traffic flow control, where on-line sensor information and actuation methods are available, the optimization-based techniques used in some of the models described in the previous section are not very well suited. Most of these models are not specifically designed for on-line control problems, such as the DTR problem discussed in this paper. They do have several drawbacks due to the fact that they were originally designed as extensions of traditional optimization-type models for off-line applications. Most of these models are not designed to take full advantage of the on-line sensor information, but they do assume a perfect knowledge of the transportation system for a specific time period. The computational requirements of these models are also very
high due to the complexity of the models and solution algorithms they use. Therefore, although
they are perfect tools for off-line planning/evaluation problems, they are not well suited for on-line
control problems. On the other hand, there exist real-time feedback control approaches that are
specifically designed for such on-line systems. The drawback of the existing linear feedback control
techniques that have previously been tried is that the system should remain in the linear region
around the equilibrium or the trajectory at all times for the controller to be valid. Since the system
is nonlinear, time varying, and contains uncertainties, feedback control laws that handle such
systems should be used.

The major motivation of this paper is to first provide a realistic DTR model specifically devel-
oped for feedback control and to use the feedback linearization technique to develop efficient and
robust feedback control laws for this nonlinear traffic flow model. Note that the technique devel-
oped in this paper is valid for single origin, single destination problems. For multi-origin, multi-
destination problems, we can use other techniques, such as nonlinear $H_{\infty}$ control, which is the
topic of research being conducted by the authors.

2.1. Preliminary considerations for using feedback control for traffic diversion

Feedback control for DTR can be an effective solution for alleviating traffic congestion during
major incidents. However, the success of such a system depends on the effective modeling of the
system as well as the design of the appropriate control law. The designer of the controller needs to
address issues such as controllability and observability of the traffic system, actuation and sensing,
robustness, and stability of the closed-loop system shown in Fig. 1.

2.1.1. Actuation and sensing. The actuation of this system can be achieved in many ways,
including variable message signs (VMS), in-vehicle guidance, and highway advisory radio (HAR).
State variables such as traffic density and average traffic speed can be sensed using various types of
traffic sensors such as inductive loops, traffic cameras, and transponders, which can be used as
sensors.

2.1.2. Controllability and observability. The designer should analyze the system before designing
the controller to determine if the system is controllable and observable. Controllability implies that
a suitable control law can be devised in order to obtain a desired response from the system. Observ-
ability implies that the system state variables can be observed from the sensed output. For instance,
if the system is not controllable, then we might decide to add more actuation infrastructure, such
as VMS or HAR, and if the system is not observable, we might add more sensors to the system.

2.1.3. Robustness and stability. The effectiveness of the control design can be measured in terms
of its robustness, stability, and transient characteristics. A robust controller will perform well even
in the presence of uncertainties in the nominal model of the system. Models representing traffic
systems cannot represent the system fully, and therefore there are uncertainties in the system that
must be dealt with. A control law should provide stability to the system and desirable transient
response. For instance, a good DTR control law would minimize the time for the system to change
from a congested state to a normal flow state.

3. SYSTEM DYNAMICS

Many researchers have studied and designed optimal open loop controllers utilizing space and
time discretized models of traffic flow (Friesz et al., 1989; Tan et al., 1993). Some researchers have

![Fig. 1. Block diagram for DTR feedback control.](image)
also designed feedback control laws using similar models (Papageorgiou, 1983, 1990; Papageorgiou and Messmer, 1991; Messmer and Papageorgiou, 1995). The reason for the popularity of these models is that there are many techniques available to deal with discretized systems. The same is also true for feedback control, and hence, in order to utilize the various linear and nonlinear (Kuo, 1987; Isidori, 1989; Mosca, 1995) control techniques available for lumped parameter systems, the distributed parameter model is space discretized (Papageorgiou, 1983). For this, the highway is subdivided into several sections, as shown in Fig. 2.

The following ordinary differential equations (ODE) with the given variable relationships can be used to model a space discretized freeway:

\[
\frac{d}{dt} \rho_i = \frac{1}{\delta_i} [q_i(t) - q_{i+1}(t) + r_i(t) - s_i(t)], \quad i = 1, 2, \ldots, n
\]

\[
q_i(t) = \rho_i(t) v_i(t)
\]

\[
v_i = v_f \left(1 - \frac{\rho_i}{\rho_{\max i}}\right)
\]

Here, \(r_i(t)\) and \(s_i(t)\) terms indicate the on-ramp and off-ramp flows, \(\rho_i\) is the density of the traffic as a function of \(x\) and time \(t\), \(q_i(t)\) is the flow at given \(x_i\), \(v_f\) is the free flow speed, and \(\rho_{\max i}\) is the jam density. Equation (1) and the output eqns (4) give the mathematical model for a highway, which can be represented in a standard nonlinear state space form for control design purposes.

\[
y_j = g_j(\rho_1, \rho_2, \ldots, \rho_n), \quad j = 1, 2, \ldots, p
\]

The standard state space form is

\[
\frac{d}{dt} \mathbf{x}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t)]
\]

\[
y(t) = \mathbf{g}[\mathbf{x}(t), \mathbf{u}(t)]
\]

\[
\mathbf{x}(0) = \mathbf{x}_0
\]

where, \(\mathbf{x} = [\rho_1, \rho_2, \ldots, \rho_n]^T\) and \(\mathbf{u}(t) = q_0(t)\).

There are various other proposed models, which are more detailed in the description of the system dynamics. The phenomenon of shock waves, which is very well represented in the PDE representation of the system, is modeled by expressing the traffic flow between two contiguous sections of the highway as the weighted sum of the traffic flows in those two sections which correspond to the densities in those two sections. A dynamic relationship instead of a static one like eqn (3) has also been proposed by Papageorgiou (1983) and used successfully. In the future, we will look into designing feedback controllers using the models which can represent spillback also. In general, however, the philosophy behind feedback control design is to use a relatively simple nominal model for control design, and the robustness of the controller should take care of the uncertainties in the system model.

4. FEEDBACK CONTROL FOR TRAFFIC

In the discretized traffic flow model, the freeway is divided into sections with aggregate traffic densities. Sensors are used to measure variables such as densities, traffic flow, and traffic average

![Fig. 2. Highway divided into sections.](image-url)
speeds in these sections, which can be used by the feedback controller to give appropriate commands to actuators like VMS and HAR.

There are essentially two ways to design controllers for such nonlinear traffic systems. One way is to design the controller by linearizing the nonlinear dynamics of the system about its equilibrium, or a trajectory; the other way is to design directly for the nonlinear system. The first way is easier, since an immense amount of literature is available describing the various design techniques, especially for linear time invariant (LTI) systems, but the results are valid only where the linearization is applicable. On the other hand, design of controllers directly for the nonlinear system is much more difficult; however, the results are usually global. Some of the linear control techniques are LQR, linear quadratic gaussian (LQG), proportional integral derivative (PID), $H_\infty$, and preview control; some of the nonlinear control techniques include describing function design, feedback linearization, sliding mode control, and nonlinear $H_\infty$. Qualitative methods such as fuzzy control and expert systems can also be utilized. Qualitative methods are usually easy to design, but are difficult to tune and analyze. Some of these issues are discussed by Papageorgiou and Messmer (1991).

In the next section, we present a user-equilibrium formulation of the DTR problem for the discretized traffic flow model given in Section 2. Then, in the following sections, three different controllers for three different versions of the same DTR problem formulation are developed using the feedback linearization technique.

4.1. DTR formulation

We first present a DTR formulation for the two alternate routes problem which is then generalized for $n$ routes case. The two routes are divided into $n_1$ and $n_2$ sections, respectively. For simplicity, we are considering the static velocity relationship, and ignoring the effect of downstream flow. Hence, the model used is

$$\frac{d}{dt} \rho_{ij} = \frac{1}{\delta_i} [q_{i,j-1}(t) - q_{i,j}(t)], \quad (i, j) = ((1, 1), (1, 2), \ldots, (1, n_1), (2, 1), (2, 2), \ldots, (2, n_2))$$

with relationships (2) and (3). The control input is given by

$$\beta(t)U(t) = q_{1,0}(t), \quad 0 \leq \beta \leq 1, \quad (1 - \beta(t))U(t) = q_{2,0}(t)$$

The flow $U(t)$ is measured as a function of time, and the splitting rate $\beta(t)$ is the control input. The output measurement could be the full state vector, i.e. vector of flows of all the sections, or a subset of that. The control problem can be stated as: find $\beta_0(t)$, the optimal $\beta(t)$, which minimizes

$$J(\beta) = \int_0^{t_f} \left[ \sum_{i=1}^{m} \chi(\rho_i) - \sum_{m+1}^{m+p} \chi(\rho_j) \right]^2 \, dt$$

where $\chi(\ldots)$ is the travel time function and $t_f$ is the final time. Note that a feedback solution is needed for the problem, not an open loop optimal control. Hence, we can either decide the structure of the feedback control, such as a PID control with constant gains, and solve numerically for the optimal values of the gains, or we can state the control objective for a standard feedback control problem, such as steady state asymptotic stability given by

$$Lt_{t\to\infty} \left[ \sum_{i=1}^{m} \chi(\rho_i) - \sum_{m+1}^{m+p} \chi(\rho_j) \right] \to 0$$

and some transient behavior characteristics such as a specified settling time or percent overshoot.

Now we formulate the same DTR formulation for a generalized case, for an $n$ alternate route problem, as follows:

_Problem:_ Find $\beta_0^i$, $i = 1, 2, \ldots, n$, which minimize
\[ j(p^i_0, i = 1, 2, \ldots, n) = t \left[ \sum_{k,p} \left( \sum_{i=1}^{n} \chi(\rho_{i,k}) - \sum_{j=1}^{n} \chi(\rho_{j,p}) \right)^2 \right] \, dt \]  

(10)

\((k = 1, 2, \ldots, n, p = 1, 2, \ldots, n,\) and the summations are taken over the total number of combinations of \(n\) and \(p\), and not permutations, so that \((k, p) = (1, 2)\) is considered the same as \((k, p) = (2, 1)\), and hence only one of these two will be in the summation), or which guarantee

\[ Lt_{t \to \infty} e \to 0 \]  

(11)

where

\[ e = \left[ \left\{ \sum_{i=1}^{n} \chi(\rho_{i,1}) - \sum_{j=1}^{n} \chi(\rho_{j,2}) \right\}, \ldots, \left\{ \sum_{i=1}^{n} \chi(\rho_{i,k}) - \sum_{j=1}^{n} \chi(\rho_{j,p}) \right\} \right] \]

with some transient behavior characteristics like a specified settling time or percent overshoot for the system

\[ \frac{d}{dt} \rho_{ij} = \frac{1}{b_i} [q_{i,j-1}(t) - q_{i,j}(t)]. \quad (i, j) = ((1, 1), \ldots, (1, n_1), (2, 1), \ldots, (2, n_2), \ldots, (n, 1), \ldots, (n, n_n)) \]

(12)

with given full and partial state observation, and input constraints

\[ \sum_{i=1}^{n} q_{i,0}(t) = U(t) \quad \text{and} \quad \sum_{i=1}^{n} p^i = 1 \]  

(13)

5. FEEDBACK LINEARIZATION TECHNIQUE

Feedback linearization is an appropriate technique for developing feedback controllers for nonlinear systems similar to the DTR model described above. The feedback linearization technique is applicable to an input affine square multiple input multiple output (MIMO) system. The details on exact nonlinear decoupling technique (feedback linearization) can be found in Isidori (1989), Slotine and Li (1991), Lighthill and Whitham (1955), and Godbole and Sastry (1995), and are briefly summarized here for the DTR application. Let us consider the following square MIMO system:

\[ \dot{x}(t) = f(x) + \sum_{i=1}^{p} g_i(x) u_i \]

\[ y_j = h_j(x) \quad j = 1, 2, \ldots, p \]  

(14)

This can be written in a compact form as

\[ \sum_{y} \dot{x}(t) = f(x) + g(x) u \]

\[ y = h(x) \]  

(15)

where \( x \in \mathbb{R}^n, f(x) : \mathbb{R}^n \to \mathbb{R}^n, g(x) : \mathbb{R}^p \to \mathbb{R}^p, u \in \mathbb{R}^p, \) and \( y \in \mathbb{R}^p \). The vector fields of \( f(x) \) and \( g(x) \) are analytic functions.

It is assumed that for the system \( \Sigma \), each output \( y_j \) has a defined relative degree \( \gamma_j \). The concept of relative degree implies that if the output is differentiated with respect to time \( \gamma_j \) times, then the control input appears in the equation. This can be succinctly represented using Lie derivatives. A definition of a Lie derivative is given below, after which the definition of relative degree in terms of Lie derivatives is stated.

**Definition (Lie derivative):** Lie derivative of a smooth scalar function \( h : \mathbb{R}^n \to \mathbb{R} \) with respect to a smooth vector field \( f : \mathbb{R}^n \to \mathbb{R}^n \) is given by \( L_f h = \frac{dh}{dt} f \). Here, \( L_f h \) denotes the Lie derivative of order zero. Higher-order Lie derivatives are given by \( L_f^j h = L_f(L_f^{j-1} h) \).
Definition (relative degree): The output $y_j$ of the system $\Sigma$ has a relative degree $\gamma_j$ if, $\exists$ an integer, s.t. $L_\theta L_r^\ell h(x) \equiv 0 \ \forall \ell < \gamma_j - 1, \ \forall i \leqslant p, \ \forall x \in U$, and $L_\theta L_r^{\gamma_j-1} h(x) \neq 0$. $U \subset \mathbb{R}^u$ which is in a given neighborhood of the equilibrium point of the system $\Sigma$. The total relative degree of the system $r$ is defined to be the sum of the relative degrees of all the output variables, i.e., $r = \sum_{j=1}^p \gamma_j$.

By successively taking the Lie derivatives of each of the output variables up to their respective relative degrees, we obtain

$$
\begin{bmatrix}
L_T^{\gamma_1} h_1(x) \\
L_T^{\gamma_2} h_2(x) \\
\vdots \\
L_T^{\gamma_p} h_p(x)
\end{bmatrix}
+ \begin{bmatrix}
L_{g_1} L_T^{\gamma_1-1} h_1(x) & \cdots & L_{g_p} L_T^{\gamma_1-1} h_1(x) \\
L_{g_1} L_T^{\gamma_2-1} h_1(x) & \cdots & L_{g_p} L_T^{\gamma_2-1} h_1(x) \\
\vdots & \ddots & \vdots \\
L_{g_1} L_T^{\gamma_p-1} h_1(x) & \cdots & L_{g_p} L_T^{\gamma_p-1} h_1(x)
\end{bmatrix} \mathbf{u}
$$

This can be written as

$$
y' = \mathbf{A}(x) + \mathbf{B}(x) \mathbf{u}
$$

where

$$
y' = \begin{bmatrix} y_1^{' \gamma_1} & y_2^{' \gamma_2} & \cdots & y_p^{' \gamma_p} \end{bmatrix}^T
$$

$$
\mathbf{A}(x) = \begin{bmatrix} L_T^{\gamma_1} h_1(x) & L_T^{\gamma_2} h_2(x) & \cdots & L_T^{\gamma_p} h_p(x) \end{bmatrix}^T
$$

$$
\mathbf{B}(x) = \begin{bmatrix}
L_{g_1} L_T^{\gamma_1-1} h_1(x) & \cdots & L_{g_p} L_T^{\gamma_1-1} h_1(x) \\
L_{g_1} L_T^{\gamma_2-1} h_1(x) & \cdots & L_{g_p} L_T^{\gamma_2-1} h_1(x) \\
\vdots & \ddots & \vdots \\
L_{g_1} L_T^{\gamma_p-1} h_1(x) & \cdots & L_{g_p} L_T^{\gamma_p-1} h_1(x)
\end{bmatrix}
$$

If the decoupling matrix $\mathbf{B}(x)$ is invertible, then we can use the feedback control law eqn (21) to obtain the decoupled dynamics eqn (22).

$$
\mathbf{u} = (\mathbf{B}(x))^{-1}[-\mathbf{A}(x) + \mathbf{v}]
$$

$$
y' = \mathbf{v}
$$

where

$$
\mathbf{v} = \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix}^T
$$

The vector $\mathbf{v}$ can be chosen to render the decoupled system eqn (17) stable with desired transient behavior. Now, if the relative degree of the system $r$ is less than the order of the system $n$, then the closed loop system should also have stable internal dynamics. In order to study that, one can define state variables $\eta_i(x)$, $i = 1, 2, \ldots, n - r$, which are independent of the state variables $r$ related to the output of the system, and are also independent of each other. The internal dynamics of the system can then be written as:

$$
\dot{\eta} = \mathbf{w}(\zeta, \eta) + \mathbf{P}(\zeta, \eta) \mathbf{u}
$$

with $(k = 1, 2, \ldots, n - r)$ and $(i = 1, 2, \ldots, p)$.
\[ w_k(\xi, \eta) = L_{\xi} \eta_k(\mathbf{x}) \]  
\[ P_{ki}(\xi, \eta) = L_{\xi} \eta_k(\mathbf{x}) \]

The feedback controller designed for eqn (17) using the feedback linearization technique should also guarantee the stability of the internal dynamics described in eqn (23). Note that for a single input case, we could use the fact that \( L_{\xi} \eta_k(\mathbf{x}) = 0 \) to choose the independent internal dynamics state variables, but for the multiple input case, this condition is not valid, unless the vectors of \( \mathbf{g} \) are involutive. Note that the control law eqn (21) has to satisfy the constraints eqn (13). In case the constraints are not satisfied, the control variables take extreme values, and the desired performance of the eigenvalues is not achieved. The moment the traffic condition changes, such that the control variables belong to the feasible set, the performance of the system comes back to the desired state. As an example, to achieve a fast response time our controller might try to over-compensate, but in reality the constraints will produce a slower rate than desired by the design; nevertheless, the system will move towards equal travel times.

6. SAMPLE PROBLEM (TWO ALTERNATE ROUTES WITH ONE SECTION)

In order to illustrate the ideas discussed above, we have designed a feedback control law for the two alternate routes problem with a single section each. The control is based on feedback linearization technique for nonlinear systems. The technique is based on defining a diffeomorphism and performing the transformation on the state variables in order to convert them into the canonical form. If the relative degree of the system is less than the system order, then the internal dynamics are studied to ensure that it is stable. The details of this technique are given in Godbole and Sastry (1995). In this problem, the system order is two and the relative degree is one.

The space discretized flow equations used for the two alternate routes are:

\[ \dot{\rho}_1 = -\frac{1}{\delta_1} \left[ \nu_{f1} \rho_1 (1 - \frac{\rho_1}{\rho_{m1}}) - \beta U \right] \]  
\[ \dot{\rho}_2 = -\frac{1}{\delta_2} \left[ \nu_{f2} \rho_2 (1 - \frac{\rho_2}{\rho_{m2}}) + \beta U - U \right] \]

We have considered a simple first order travel time function, which is obtained by dividing the length of a section by the average velocity of the vehicles on it. According to that, the travel time can be calculated as

\[ \chi_1(k) = \frac{1}{v_{f1}} \left( 1 - \frac{\rho_1}{\rho_{m1}} \right) \]  
\[ \chi_2(k) = \frac{1}{v_{f2}} \left( 1 - \frac{\rho_2}{\rho_{m2}} \right) \]

where \( d_1 \) and \( d_2 \) are section lengths, \( v_{f1} \) and \( v_{f2} \) are the free flow speeds of each section, and \( \rho_{m1} \) and \( \rho_{m2} \) are the maximum (jam) densities of each section. Since we need to equate the travel times according to the UE DTR formulation discussed in the previous section, we take the new transformed state variable \( y \) as the difference in travel times. Differentiating the equation representing \( y \) in terms of the state variables introduces the input split factor into the dynamic equation. Therefore, that transformed equation can be used to design the input that cancels the nonlinearities of the system and introduces a design input \( v \), which can be used to place the poles of the error equation for asymptotic stability. These steps are shown below.

The variable \( y \) is equal to the difference in the travel time on the two sections.

\[ y = \frac{k}{(k_2 - \rho_1)} - \frac{k_3}{(k_4 - \rho_2)} \]
where
\[ k_1 = \frac{d_1 \rho_{m_1}}{v_{f_1}}, \quad k_2 = \rho_{m_1}, \quad k_3 = \frac{d_2 \rho_{m_2}}{v_{f_2}}, \quad k_4 = \rho_{m_2} \]

This equation can be differentiated with respect to time to give the travel time difference dynamics.

\[ \dot{y} = \frac{k_1 \dot{\rho}_1}{(k_2 - \rho_1)^2} - \frac{k_3 \dot{\rho}_2}{(k_4 - \rho_2)^2} \]  

(31)

By substituting eqns (26) and (27) in (31), we obtain

\[ \dot{y} = -\frac{k_1 \left( v_{f_1} \frac{\rho_1 - \rho_{m_1}}{\rho_{m_1}} - \beta U \right)}{\delta_1 (k_2 - \rho_1)^2} + \frac{k_3 \left( v_{f_2} \frac{\rho_2 - \rho_{m_2}}{\rho_{m_2}} + \beta U \right)}{\delta_2 (k_4 - \rho_2)^2} \]

(32)

This equation can be rewritten in the following form:

\[ \dot{y} = F + G \beta \]

(33)

where

\[ F = \left[ -\frac{k_1 v_{f_1} \rho_1}{\delta_1 (k_2 - \rho_1)^2} \left( 1 - \frac{\rho_1}{\rho_{m_1}} \right) + \frac{k_3}{\delta_2 (k_4 - \rho_2)^2} \left( 1 - \frac{\rho_2}{\rho_{m_2}} \right) v_{f_2} \rho_2 - U \right] \]

(34)

\[ G = \left( \frac{k_1}{\delta_1 (k_2 - \rho_1)^2} + \frac{k_3}{\delta_2 (k_4 - \rho_2)^2} \right) U \]

(35)

Hence, a feedback linearization control law can be designed to cancel the nonlinearities and provide the desired error dynamics. The feedback control law given in eqn (21) is used,

\[ \beta = G^{-1}(-F + v) \]

which gives the closed loop dynamics as

\[ y = v \]

(37)

As was mentioned earlier, since the relative degree of the system is one, and the system order is two, we need to test the stability or boundedness of the second transformed state variable given by

\[ \eta = \delta_1 \rho_1 + \delta_2 \rho_2 \]

(38)

The state variable \( \eta \) is bounded since the densities on the sections cannot exceed the corresponding jam densities. This assumes that no traffic from ramps enters a section with jam density, and also that at the node, the traffic flow into the sections is zero if jam density is reached. This is a reasonable assumption since measured traffic density will never become higher than the jam density.

\[ \eta \leq \delta_1 \rho_{m_1} + \delta_2 \rho_{m_2} \]

(39)

and hence the overall system is exponentially stable \( (y \to 0) \) if we choose \( v = -Ky, K > 0 \), and \( y \) asymptotically goes to zero as \( y(t) = y(0) e^{-Kt} \). This implies that when a splitting value based on (36) is utilized, the difference in travel time of two alternate routes will go to zero at an exponential rate. Hence, the closed loop traffic system controlled by the proposed feedback linearization law is exponentially stable and has desired transient behavior. Note that if input saturation occurs, the rate of convergence cannot be guaranteed.
7. SAMPLE PROBLEM (TWO ALTERNATE ROUTES WITH TWO SECTIONS)

Now, we extend the above problem to a case with two sections and follow the same steps for designing a new controller for this extended system. The space discretized flow equations used for the two alternate routes are:

\[
\dot{\rho}_{11} = -\frac{1}{\delta_{11}} \left[ \nu_{11}\rho_{11}\left(1 - \frac{\rho_{11}}{\rho_{n11}}\right) - \beta U \right] \\
\dot{\rho}_{12} = -\frac{1}{\delta_{12}} \left[ \nu_{12}\rho_{12}\left(1 - \frac{\rho_{12}}{\rho_{n12}}\right) - \nu_{11}\rho_{11}\left(1 - \frac{\rho_{11}}{\rho_{n11}}\right) \right] \\
\dot{\rho}_{21} = -\frac{1}{\delta_{21}} \left[ \nu_{21}\rho_{21}\left(1 - \frac{\rho_{21}}{\rho_{n21}}\right) + \beta U - U \right] \\
\dot{\rho}_{22} = -\frac{1}{\delta_{12}} \left[ \nu_{22}\rho_{22}\left(1 - \frac{\rho_{22}}{\rho_{n22}}\right) - \nu_{21}\rho_{21}\left(1 - \frac{\rho_{21}}{\rho_{n21}}\right) \right]
\]

We have considered a simple first order travel time function, which is obtained by dividing the length of a section by the average velocity of vehicles on it. According to that, we approximate travel time as

\[
\chi_1(t) = \frac{d_{11}}{\nu_{11}(1 - \frac{\rho_{11}}{\rho_{n11}})} + \frac{d_{12}}{\nu_{12}(1 - \frac{\rho_{12}}{\rho_{n12}})} \\
\chi_2(t) = \frac{d_{21}}{\nu_{21}(1 - \frac{\rho_{21}}{\rho_{n21}})} + \frac{d_{22}}{\nu_{22}(1 - \frac{\rho_{22}}{\rho_{n22}})}
\]

where \( d_1 \) and \( d_2 \) are section lengths, \( \nu_1 \) and \( \nu_2 \) are the free flow speeds of each section, and \( \rho_{n1} \) and \( \rho_{n2} \) are the maximum (jam) densities of each section.

The system can be written in the standard nonlinear input affine form

\[
\dot{x}(t) = f(x, t) + g(x, t)u(t) \\
y(t) = h(x, t)
\]

where

\[
x = [\rho_{11} \rho_{12} \rho_{21} \rho_{22}], \quad u(t) = \beta
\]

\[
f = \begin{bmatrix}
-\frac{1}{\delta_{11}} \left[ \nu_{11}\rho_{11}\left(1 - \frac{\rho_{11}}{\rho_{n11}}\right) \right] \\
-\frac{1}{\delta_{12}} \left[ \nu_{12}\rho_{12}\left(1 - \frac{\rho_{12}}{\rho_{n12}}\right) - \nu_{11}\rho_{11}\left(1 - \frac{\rho_{11}}{\rho_{n11}}\right) \right] \\
-\frac{1}{\delta_{21}} \left[ \nu_{21}\rho_{21}\left(1 - \frac{\rho_{21}}{\rho_{n21}}\right) - \nu_{11}\rho_{11}\left(1 - \frac{\rho_{11}}{\rho_{n11}}\right) \right] \\
-\frac{1}{\delta_{12}} \left[ \nu_{22}\rho_{22}\left(1 - \frac{\rho_{22}}{\rho_{n22}}\right) - \nu_{21}\rho_{21}\left(1 - \frac{\rho_{21}}{\rho_{n21}}\right) \right]
\end{bmatrix}
\]

\[
g = \begin{bmatrix}
U \\
O \\
-U \\
O
\end{bmatrix}
\]
Since we need to equate the travel times on alternate routes according to the UE DTR problem formulation presented in the previous section, we take the new transformed state variable $y$ as the difference in travel times. Differentiating the equation representing $y$ in terms of the state variables introduces the input split factor into the dynamic equation. Therefore, that transformed equation can be used to design the input that cancels the nonlinearities of the system and introduces a design input $v$, which can be used to place the poles of the error equation for asymptotic stability. These steps are shown below.

The variable $y$ is equal to the difference in the travel time on the two sections.

$$ y(t) = \left[ \frac{d_{11}}{v_{11}(1 - \rho_{11}^{(12)})} + \frac{d_{12}}{v_{12}(1 - \rho_{12}^{(12)})} \right] - \left[ \frac{d_{21}}{v_{21}(1 - \rho_{21}^{(22)})} + \frac{d_{22}}{v_{22}(1 - \rho_{22}^{(22)})} \right] $$  \hspace{1cm} (50)

This equation can be differentiated with respect to time to give the travel time difference dynamics:

$$ \dot{y} = \frac{k_1 \dot{\rho}_{11}}{(k_2 - \rho_{11})^2} + \frac{k_3 \dot{\rho}_{12}}{(k_4 - \rho_{12})^2} - \frac{k_5 \dot{\rho}_{21}}{(k_6 - \rho_{21})^2} - \frac{k_7 \dot{\rho}_{22}}{(k_8 - \rho_{22})^2} \hspace{1cm} (51) $$

By substituting eqns (40)–(43) in (51), we obtain

$$ \dot{y} = \frac{k_1 \frac{1}{\gamma_1} \left[ v_{11} \rho_{11} \left( 1 - \frac{\rho_{11}}{\rho_{11}^{(12)}} \right) - \beta U \right]}{(k_2 - \rho_{11})^2} - \frac{k_3 \frac{1}{\gamma_2} \left[ v_{12} \rho_{12} \left( 1 - \frac{\rho_{11}}{\rho_{11}^{(12)}} \right) - v_{11} \rho_{11} \left( 1 - \frac{\rho_{11}}{\rho_{11}^{(12)}} \right) \right]}{(k_4 - \rho_{12})^2} + $$

$$ + \frac{k_5 \frac{1}{\gamma_3} \left[ v_{21} \rho_{21} \left( 1 - \frac{\rho_{11}}{\rho_{11}^{(22)}} \right) + \beta U - U \right]}{(k_6 - \rho_{21})^2} + \frac{k_7 \frac{1}{\gamma_4} \left[ v_{22} \rho_{22} \left( 1 - \frac{\rho_{11}}{\rho_{11}^{(22)}} \right) - v_{21} \rho_{21} \left( 1 - \frac{\rho_{11}}{\rho_{11}^{(22)}} \right) \right]}{(k_8 - \rho_{22})^2} $$  \hspace{1cm} (52)

This equation can be rewritten in the following form:

$$ \dot{y} = F + G \beta $$  \hspace{1cm} (53)

where

$$ F = \frac{k_1 \frac{1}{\gamma_1} \left[ v_{11} \rho_{11} \left( 1 - \frac{\rho_{11}}{\rho_{11}^{(12)}} \right) \right]}{(k_2 - \rho_{11})^2} - \frac{k_3 \frac{1}{\gamma_2} \left[ v_{12} \rho_{12} \left( 1 - \frac{\rho_{11}}{\rho_{11}^{(12)}} \right) - v_{11} \rho_{11} \left( 1 - \frac{\rho_{11}}{\rho_{11}^{(12)}} \right) \right]}{(k_4 - \rho_{12})^2} + $$

$$ + \frac{k_5 \frac{1}{\gamma_3} \left[ v_{21} \rho_{21} \left( 1 - \frac{\rho_{11}}{\rho_{11}^{(22)}} \right) \right]}{(k_6 - \rho_{21})^2} + \frac{k_7 \frac{1}{\gamma_4} \left[ v_{22} \rho_{22} \left( 1 - \frac{\rho_{11}}{\rho_{11}^{(22)}} \right) - v_{21} \rho_{21} \left( 1 - \frac{\rho_{11}}{\rho_{11}^{(22)}} \right) \right]}{(k_8 - \rho_{22})^2} $$  \hspace{1cm} (54)

$$ G = \left[ \frac{k_1 \frac{1}{\gamma_1}}{(k_2 - \rho_{11})^2} + \frac{k_3 \frac{1}{\gamma_2}}{(k_4 - \rho_{12})^2} \right] U $$  \hspace{1cm} (55)

Hence, a feedback linearization control law similar to the one given by eqn (21) can be designed to cancel the nonlinearities and provide the desired error dynamics. The law used is

$$ \beta = G^{-1}(-F + v) $$  \hspace{1cm} (56)

which gives the closed loop dynamics as

$$ y = v $$  \hspace{1cm} (57)
The relative degree of the system is one, and the system order is four, and we need to test the stability or boundedness of the three internal states. Now our task is to obtain the other three independent state variables. Since this is a single input system, to obtain these state variables, we should satisfy $Lg\eta_k(x) = 0$ as follows:

\[ \frac{\partial \eta_i}{\partial \rho_{ij}}, i = 1, 2, 3 \]

\[ \eta_1 = \rho_{12} \]
\[ \eta_2 = \rho_{22} \]
\[ \eta_3 = \rho_{11} + \rho_{21} \]  

(58)

The state variable $\eta$ is bounded since the densities on the sections cannot exceed the corresponding jam densities. We use the same argument as for the previous example for this claim.

\[ \eta_1 \leq \rho_{n12} \]
\[ \eta_2 \leq \rho_{n22} \]
\[ \eta_3 \leq \rho_{11} + \rho_{12} \]  

(59)

and hence the overall system is exponentially stable ($y \to 0$) if we choose $\nu = -Ky$, $K > 0$, and $y$ asymptotically goes to zero as $y(t) = y(0)e^{-Kt}$. This implies that when a splitting value based on eqn (56) is utilized, the difference in travel time of two alternate routes will go to zero at an exponential rate. Hence, the closed loop traffic system controlled by the proposed feedback linearization law is exponentially stable and has desired transient behavior.

8. SOLUTION FOR THE ONE-ORIGIN, ONE-DESTINATION CASE WITH MULTIPLE ROUTES WITH MULTIPLE SECTIONS.

In this section, we give a generalized solution for the $n$ alternate route DTR problem described in Section 3.1. The space discretized flow equations used for the $n$ alternate routes and $n$ sections are given by eqns (12) and (13). The number of sections for each alternate route $I$ is denoted by $n_I$. We are considering full state observation, which is used for estimating (sensing) the travel times on the various alternate routes. The dynamics can be written as

\[ \dot{\rho}_{ij} \frac{1}{\delta_{ij}} \left[ \nu_{ij-1} \rho_{ij-1} (1 - \frac{\rho_{ij-1}}{\rho_{mij-1}}) - \nu_{ij-1} \rho_{ij} (1 - \frac{\rho_{ij}}{\rho_{mij}}) \right] \]  

(60)

when $(i, j) = ((1, 2), \ldots, (1, n_1), (2, 2), \ldots, (2, n_2), \ldots, (n, 2), \ldots, (n, n_n))$

\[ \dot{\rho}_{ij} \frac{1}{\delta_{ij}} \left[ \beta_{ij} U - \nu_{ij} \rho_{ij} (1 - \frac{\rho_{ij}}{\rho_{mij}}) \right] \]  

(61)

when $(i, j) = ((1, 1), (2, 1), \ldots, (n, 1))$

We have considered a simple first order travel time function, which is obtained by dividing the length of a section by the average velocity of vehicles on it. According to that, we approximate travel time for a route as

\[ \chi_i(t) = \sum_{j=1}^{n} \frac{d_{ij}}{\nu_{ij} (1 - \frac{\rho_{ij}}{\rho_{mij}})} \]  

(62)

The system can be written in the standard nonlinear input affine form:

\[ \dot{x}(t) = f(x, t) + g(x, t)u(t) \]
\[ y(t) = h(x, t) \]  

(63)
where
\[ \mathbf{x} = [\rho_1 \ldots \rho_{n1} \ldots \rho_n \ldots \rho_{nn}]', \mathbf{u}(t) = [\beta_1 \ldots \beta_{n-1}] \] (64)

The output vector is denoted by \( \mathbf{y} \), and is given by:
\[ \mathbf{y} = [y_1 y_2 \ldots y_i \ldots y_{n-1}] \] (65)

where
\[ y_i = \chi_{i+1}(t) - \chi_i(t) \] (66)

This equation can be differentiated with respect to time to give the travel time difference dynamics.
\[ \dot{y}_i = \sum_{j=1}^{n+1} \frac{k_{i+1,j} \dot{\rho}_{i+1,j}}{k_{i+1,j}^2 \left(1 - \frac{\rho_{i+1,j}}{\rho_{i+1}}\right)^2} - \sum_{j=1}^{n} \frac{k_{ij} \dot{\rho}_{ij}}{k_{ij}^2 \left(1 - \frac{\rho_{ij}}{\rho_{ij}}\right)^2} \] (67)

where \( k_{ij} \) denotes a constant \( p = 1 \) or \( 2 \) that belongs to section \( j \) of route \( i \), similar to the constant \( k \) described for eqn (30). The system (67) is in the form of eqn (16) and can be represented in the form of eqn (17) by determining the values of \( A(x) \) and \( B(x) \). The control law (21) provides us with user equilibrium for the DTR problem.

The input appears in all the output equations after differentiating them one time. Hence, the relative degree of the system is \( n - 1 \). The order of the system is \( \sum_{i=1}^{n} n_i \). Since the densities on the sections are bounded by jam density values, the independent state variables \( \eta_i, i = 1, 2, \ldots, \sum_{i=1}^{n} n_i + 1 - n \) are also bounded.

9. SIMULATION

Several simulation studies are performed to demonstrate the utilization of the feedback linearization technique presented in this paper. The test network, which consists of two alternate routes, is shown in Fig. 3. Three different simulation scenarios that were chosen are:

(a) model without any parametric uncertainties and full user compliance,
(b) model with parametric uncertainties and full user compliance,
(c) model with parametric uncertainties and partial user compliance.

The input function is assumed to be a sinusoidal function which reaches a pre-defined peak value and then settles at a constant value for the rest of the simulation period. This function emulates the peak hour demand that reaches its maximum value at a certain time, and then settles at a constant value when the peak period is over. In this specific simulation study, the peak period is assumed to be 1 h.

9.1. Scenario 1: model with full user compliance and without any uncertainties

In this scenario, the controller has perfect knowledge of the parameters of the traffic model. In addition, full compliance of the users to the diversion commands is assumed by the controller. The system dynamics model also simulates full compliance of the users to the controller \( \pm s \) diversion.
Fig. 4. (a) Differences in travel times (s) for scenario 1; (b) split factors for scenario 1.

Fig. 5. (a) Differences in travel times (s) for scenario 2; (b) split factors for scenario 2.
commands. In this case, since the controller has the complete knowledge of the system dynamics, it is able to perform exact cancellation of the system nonlinearities and attain exponential error convergence. This result is shown in Fig. 4(a). Figure 4(b) shows split factors.

9.2. Scenario 2: model with full user compliance and uncertainties

In this scenario, the controller does not have perfect knowledge of the parameters of the traffic model. In order to simulate the effects of uncertainties, ±30% errors have been assumed. However, full compliance of the users to the diversion commands is assumed by the controller and is also simulated by the system model. In this case, since the controller does not have complete knowledge of the system dynamics, it is not able to perform exact cancellation of the system nonlinearities and attain exponential error convergence. However, as can be seen in Fig. 5(a) and (b), the results obtained by using this controller, even with such relatively large parametric uncertainty, are highly encouraging.

9.3. Scenario 3: model with partial user compliance and uncertainties

In this scenario, we assume both partial user compliance (80%), and the existence of parametric uncertainty in the model. As can be seen in Fig. 6(a) and (b), the fluctuations of differences in travel are much higher than the previous scenarios, and it takes the controller a longer time to attain error convergence. However, even with partial user compliance and fairly large parametric uncertainties, the system stabilizes and the differences in travel times asymptotically converge at a desirable rate.

9.4. Scenario 4: model with partial user compliance and uncertainties, and a linear PI controller

In this scenario, we assume both partial user compliance (80%), and the existence of parametric uncertainty in the model, and we use a linear PI (proportional, integral) controller. As is evident here, the performance of this controller is inferior to the feedback linearization controller [Fig. 7(a) and (b)].
Fig. 7. (a) Differences in travel times (s) for scenario 4; (b) split factors for scenario 4.

Fig. 8. (a) Differences in travel times (s) for scenario 5; (b) split factors for scenario 5.
9.5. Scenario 5: model with partial user compliance, uncertainties, and dynamic velocity relationship

In this scenario, we assume both partial user compliance (80%), and the existence of parametric uncertainty in the model. We also consider a dynamic relationship for velocity to represent the shock wave dynamics in the model. The results of using the feedback linearization to uncertainties obtained by under-modeling of this kind are also satisfactory [Fig. 8(a) and (b)].

In all the scenarios, there is no congestion created in the two routes. In general, the control algorithm cannot prevent congestion if there is a very large inflow of traffic. Consider the situation where the inflow traffic is so high that it produces traffic which is greater than the overall capacity of the two routes; then the split factor control cannot avoid congestion. In that case, some traffic from the inflow itself would have to be diverted.

9.6. Simulation environment

The simulation environment we used is SIMNON (Elmqvist, 1975), which is a special programming language developed in the Lund Institute of Technology, Sweden, for simulating dynamic systems described as ordinary differential equations, as difference equations, or as combinations of both. This program is available in DOS and Windows environments.

10. DEPLOYMENT ISSUES

We are making plans for deploying this strategy at the Suffolk area in Virginia (Fig. 9).

The morning rush hour flow is from point A to B, and the evening rush hour in the other direction. Although this problem is really a network level problem with multiple nodes and routes, the majority of the traffic flows only in two alternate routes. The feedback linearization technique can be applied to this problem for calculating the desired split rates, and then another algorithm could be designed to come up with the means to control the various ways to achieve that. For instance, we would use variable message signs and highway advisory radio. What messages to put on these systems is part of a separate research topic, but the results of those could be combined with the feedback control design to achieve an overall control system. The feedback control law is designed with the nominal model, and the robustness properties of the controller are used to handle the uncertainties in the nominal model. We are currently designing controllers for network level problems using feedback control. Kachroo et al. (1997a,1998) and Kachroo and Özbay (1997) deal with formulations and control solutions for those problems.

11. CONCLUSIONS

In this paper, we have addressed the real-time traffic control problem for point diversion. A feedback model is developed for control purposes, and feedback linearization technique is used to design this feedback controller. First, the simplest case, with two alternate routes consisting of a single section each, is studied and a feedback controller using feedback linearization technique is
developed. Second, the case with two alternate routes with two discrete sections is analyzed, and a feedback controller using feedback linearization technique is also developed. Finally, the general case with multiple alternate routes divided into multiple sections is analyzed, and a general solution is proposed. To illustrate the above models, simulation runs are performed for three different scenarios for a network topology of two alternate routes. The feedback controller developed for this test network performed fairly well for all three scenarios. An important finding of this simulation study was the robustness of the controller even for situations where parametric uncertainties exist and partial user compliance is employed. Therefore, we can conclude that feedback linearization is an effective method for designing real-time traffic control systems.

REFERENCES


