

## Chapter 5

### Time Value of Money

## Topics

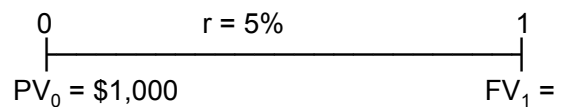
1. Future Value of a Lump Sum
2. Present Value of a Lump Sum
3. Future Value of Cash Flow Streams
4. Present Value of Cash Flow Streams
5. Perpetuities
6. Uneven Series of Cash Flows
7. Other Compounding Periods
8. Effective Annual Rates (EAR)
9. Amortized Loans

## Future Value - Compounding

Future Value - What is something worth in the future at a rate of interest.

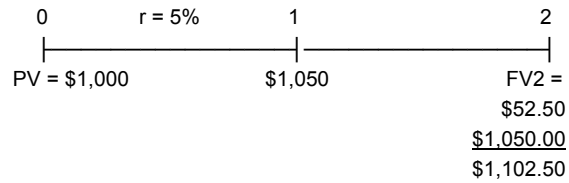
ex. \$1,000 deposited into a bank today and left in the account for 1 year earning interest at 5%

## Future Value



$$\begin{aligned} \text{FV}_1 &= \text{PV} + \text{Interest} \\ &= \$1,000 + \text{Interest} (r = \$1,000 * .05) \\ &= \$1,000 + \$50 \\ &= \underline{\$1050} \end{aligned}$$

## How about in 2 years?



## Compounding

Compounding = interest on interest

$$FV_1 = PV + (PV \times r) = PV (1 + r)$$

$$FV_2 = PV (1 + r) (1 + r) = PV (1 + r)^2$$

General form of equation =

$$FV_n = PV (1 + r)^n$$

## Future Values: General Formula

$$FV = PV(1 + r)^n$$

FV = future value

PV = present value

r = period interest rate, expressed as a decimal

n = number of periods

## FV Practice

What is the future value (FV) of \$100 if it is invested 1 year at 10%?

$$\begin{aligned} FV_1 &= PV (1 + r)^1 \\ &= \$100 (1.10)^1 \\ &= \$110 \end{aligned}$$

## FV Practice

What is the future value (FV) of \$100 if it is invested 2 years at 10%?

$$\begin{aligned}FV_2 &= PV ( 1 + r )^2 \\ &= \$100 (1.10)^2 \\ &= \$121\end{aligned}$$

## FV of \$100 at 10%



Growth of \$100 original amount at 10% per year. Blue shaded area represents the portion of the total that results from compounding of interest.

## Solving for FV

What is the FV of \$100 if invested for 3 years earning 10% interest?

Solve with the equation

$$\begin{aligned}FV_n &= PV ( 1 + r )^n \\ &= \$100 ( 1.10 )^3 \\ &= \$133.10\end{aligned}$$

## Solving for FV

Solve with the calculator

$$PV = 100$$

$$N = 3$$

$$I/Y = 10\%$$

$$FV = \underline{-133.10}$$

## Calculator Rules

1. Clear Register
2. Set to 1 period per year
3. Set proper mode (Begin or End)

## Calculator Keys

Texas Instruments BA-II Plus

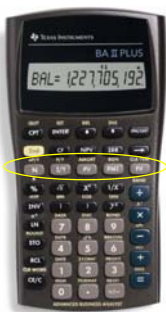
FV = future value

PV = present value

I/Y = period interest rate

N = number of periods

## Texas Instruments BA-II Plus



The highlighted row of keys is the TVM register. Solve for FV, PV, PMT, N, or I/Y

N	Number of periods
I/Y	Interest rate per period
PV	Present value
PMT	Payment per period
FV	Future value

CLR TVM: Clears all keys in the TVM register

## Using The TI BAII+ Calculator

Focus on 3rd Row of keys



## Calculator Keys

Hewlett-Packard 10B

FV = future value

PV = present value

I/YR = period interest rate

N = number of periods

## Practice

What is the future value (FV) of \$1,000 if it is invested 20 years at 5%?

PV = \$1,000

I/Y = 5%

N = 20

$FV_{20} = \$2,653.30$

## Practice

What is the future value (FV) of \$1,000 if it is invested 20 years at 10%?

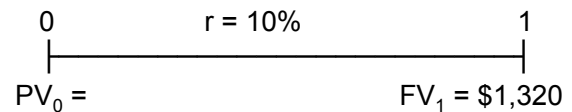
PV = \$1,000

I/Y = 10%

N = 20

$FV_{20} = \$6,727.50$

## Present Value - Discounting



### Solve with the equation

$$\begin{aligned}PV_0 &= FV_n / (1 + r)^n \\ &= \$1,320 / (1.10)^1 \\ &= \$1,200\end{aligned}$$

### Solve with the calculator

$$\begin{aligned}PV &= \$1,320 \\ I/Y &= 10\% \\ N &= 1\end{aligned}$$

$$FV_1 = \$1,200.00$$

### Problem

Oscar Wright of Carson City, Nevada, while combing through his great-grandmother's trunk of remembrances, found a State of Nevada Bond that was issued the day Nevada became the 36th state on October 31, 1864. This State of Nevada bond has a face value of \$100 and a stated interest rate of 22%. Why do you think the interest rate is so high? If the State of Nevada agrees to pay Oscar the face value of this bond plus interest through October 31 of last year, how much will he receive?

### Answer

$$\begin{aligned}PV &= \$100 \\ I/Y &= 22\% \\ N &= 147\end{aligned}$$

$$FV_{147} = \$4.953 \times 10^{14}$$

## Doubling Money

How long will it take a \$1,000 investment to double if it is invested in an account paying 8% interest?

$$PV = -\$1,000$$

$$FV = +\$2,000$$

$$I/Y = 8\%$$

$$N = 9.0065 \text{ yrs}$$

## Doubling Money

The "Rule of 72". A method to approximate when solving for n or r.

$$n = 72 / r$$

or

$$r = 72 / n$$

## Problem

A recent advertisement in the financial section of a magazine carried the following claim: "Invest your money with us at 6 percent, compounded annually, and we guarantee to double your money sooner than you imagine." Ignoring taxes, approximately how long would it take to double your money at a nominal rate of 6 percent, compounded annually? How long if the rate is 3%? How much if the rate is 18%?

## Answer @6%

$$PV = -\$1$$

$$FV = +\$2$$

$$I/Y = 6\%$$

$$N = 11.8957 \text{ yrs}$$

### Answer @3%

PV = -\$1  
FV = +\$2  
I/Y = 3%

**N = 23.4498 yrs**

### Answer @18%

PV = -\$1  
FV = +\$2  
I/Y = 18%

**N = 4.1878 yrs**

### Problem

If you had invested \$10,000 in Fidelity's Magellan fund when Peter Lynch became the fund manager on 5/31/77 your money would have grown to \$233,139.61 by the time Lynch retired on 5/30/92. What is the annual rate of return on your investment?

### Answer

PV = -\$10,000  
FV = +\$233,139.61  
N = 15  
**I/Y = 23.36%**

## Annuities

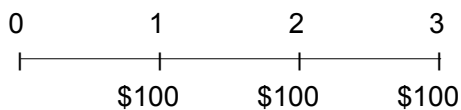
- ◆ **An Annuity** represents a series of equal payments (or receipts) occurring over a specified number of equidistant periods.
- Examples of Annuities Include:
  - Student Loan Payments
  - Car Loan Payments
  - Insurance Premiums
  - Mortgage Payments
  - Retirement Savings

## *Types of Annuities*

- ◆ **An Annuity** represents a series of equal payments (or receipts) occurring over a finite period of time.
- **Ordinary Annuity:** Payments or receipts occur at the **end** of each period.
- **Annuity Due:** Payments or receipts occur at the **beginning** of each period.

### Future Value of an Ordinary Annuity

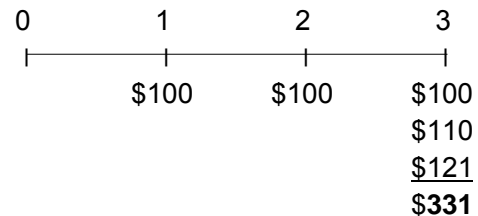
A payment of \$100/year for 3 years at 10%, with the first payment one year from today



$FV_3 =$

### FV of an Ordinary Annuity

Find the  $FV_3$  of each \$100 at 10%



## FV of an Ordinary Annuity

Equation

$$\begin{aligned}
 FV &= PMT * \frac{(1+r)^n - 1}{r} \\
 &= \$100 * \frac{(1.10)^3 - 1}{.10} \\
 &= \$100 * 3.3100 = \$331.00
 \end{aligned}$$

## FV of an Ordinary Annuity

Calculator

PMT=100

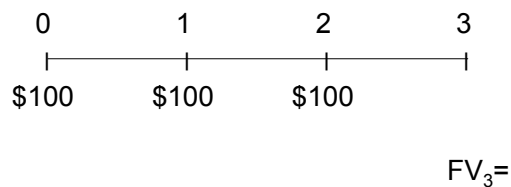
N=3

I/Y=10%

**FV = \$331.00**

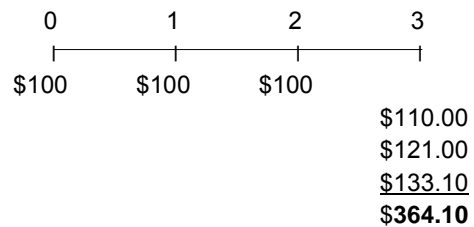
## Future Value of an Annuity Due

A payment of \$100/year for 3 years at 10%,  
with the first payment today



## FV of an Annuity Due

Find the  $FV_3$  of each \$100 at 10%



### FV of an Annuity Due

Equation

$$\begin{aligned}
 FV &= PMT \times \frac{(1+r)^n - 1}{r} \times (1+r) \\
 &= \$100 \times \frac{(1.10)^3 - 1}{.10} \times (1.10) \\
 &= \$100 \times 3.6410 = \$364.10
 \end{aligned}$$

### FV of an Annuity Due

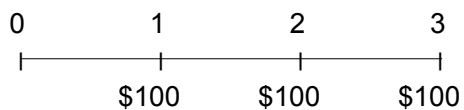
Calculator

PMT=100  
 N=3  
 I/Y=10%  
 Beg Mode

**FV = \$364.10**

### Present Value of an Ordinary Annuity

A payment of \$100/year for 3 years at 10%,  
 with the first payment one year from today



PV<sub>0</sub>=

### PV of an Ordinary Annuity

Find the PV<sub>0</sub> of each \$100 at 10%

0	1	2	3
----- ----- -----			
	\$100	\$100	\$100
\$ 90.91			
\$ 82.64			
<u>\$ 75.13</u>			
<b>\$248.68</b>			

### PV of an Ordinary Annuity

Equation

$$\begin{aligned}
 PV &= PMT \times \left[ \frac{1 - 1 / (1 + r)^n}{r} \right] \\
 &= PMT \times \left[ \frac{1 - 1 / (1.10)^3}{.10} \right] \\
 &= \$100 \times 2.4869 = \$248.69
 \end{aligned}$$

### PV of an Ordinary Annuity

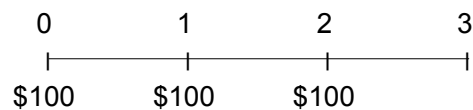
Calculator

PMT=100  
N=3  
I/Y=10%

**PV = \$248.6852**

### Present Value of an Annuity Due

A payment of \$100/year for 3 years at 10%, with the first payment today



PV<sub>0</sub>=

### PV of an Annuity Due

Find the PV<sub>0</sub> of each \$100 at 10%

0	1	2	3
\$100	\$100	\$100	
\$ 90.91			
<u>\$ 82.64</u>			
<b>\$273.55</b>			

## PV of an Annuity Due

Equation

$$\begin{aligned}PV &= PMT \times \left[ \frac{1 - 1 / (1 + r)^n}{r} \right] \times (1 + r) \\ &= 100 \times \left[ \frac{1 - 1 / (1.10)^3}{.10} \right] \times (1.10) \\ &= \$100 \times 2.4869 \times 1.10 = \$273.56\end{aligned}$$

## PV of an Annuity Due

Calculator

BGN  
PMT=100  
N=3  
I/Y=10%

**PV = \$248.6852**

## Problem

Assume you start investing for your retirement by opening a Roth IRA and depositing money into a mutual fund on a yearly basis. These deposits consist of \$3,000 per year and are in the form of a 40-year annuity due. Assume this fund earns 11% per year over these 40 years. How much money will be in your retirement account the day you retire? When you retire in year 40 you move your IRA nest egg into a safer account earning 5% per year. Assume you wish to withdraw an equal annual amount for 25 years as an ordinary annuity until all the money is gone. How much can you withdraw every year?

## Problem

You decide that when you retire 50 years from now, you will need \$200,000 a year to live comfortably for the next 20 years (you receive these payments starting 51 years from now). If money is deposited into an account with a contract rate of interest of 10 percent, how much will you need to save every year (deposit in an account) for the next 50 years (assume you make 50 payments). Assume the first savings deposit starts today and money remains in this account paying the same interest rate until all funds are paid out.

## Perpetuities

- Infinite series of payments
- An annuity that goes on forever

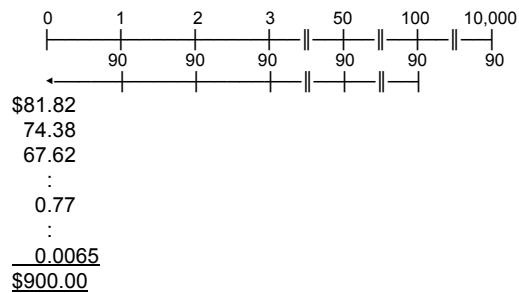
$$PV = \frac{PMT}{r}$$

## Example

A bond is contracted to make a \$90 payment per year, the first payment is one year from today, there is no maturity date, and the market rate of interest is 10%. What is the PV?

$$PV = \frac{PMT}{r} = \frac{\$90}{.10} = \$900$$

## Example



## Present Value of a Growing Perpetuity

$$PV = \frac{CF}{r - g}$$

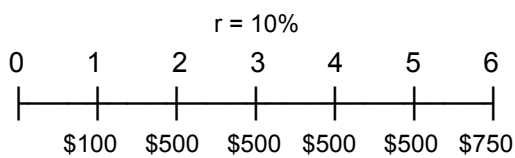
### Problem

The Andre Aggasi Foundation wishes to endow the Las Vegas Animal Shelter with a annual gift that will grow with inflation. The amount given the first year is \$10,000 and inflation is expected to be 3% per year. If the discount rate is 10%, what is the present value of this endowment?

### Problem

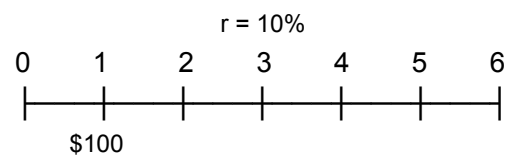
If instead inflation is estimated to be 5% per year and the discount rate remains at 10%, what is the present value of this endowment?

### Uneven Series of Cash Flows



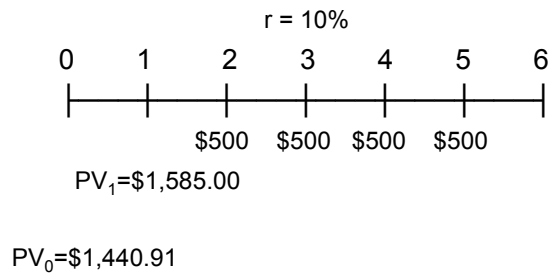
$PV_0 = ?$

### Uneven Series of Cash Flows

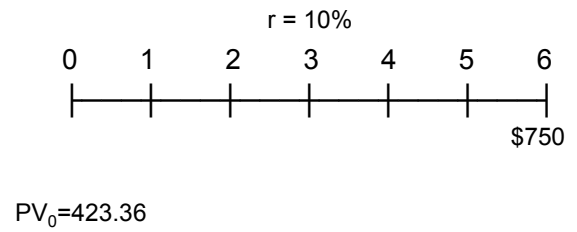


$PV_0 = \$90.91$

### Uneven Series of Cash Flows



### Uneven Series of Cash Flows



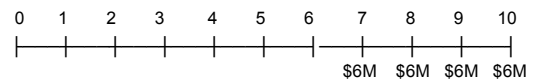
### Uneven Series of Cash Flows

Total  $PV_0 =$

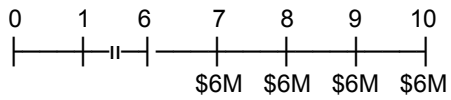
\$	90.91
+	\$1,440.91
+	<u>\$ 423.36</u>
=	<b>\$1,955.18</b>

### Complex TVM

Assuming a discount rate of 9%, what is the value of these cash flows in year 6? Year 7?, Year 10?, Year 11?



### Value in Year 6



END  
PMT=\$6M  
r = 9%  
n = 4  
 $PV_6 = \$19.44M$

### Value in Year 7



BGN  
PMT=\$6M  
r = 9%  
n = 4  
 $PV_7 = \$21.19M$

### Value in Year 10



END  
PMT=\$6M  
r = 9%  
n = 4  
 $FV_{10} = \$27.44M$

### Value in Year 11



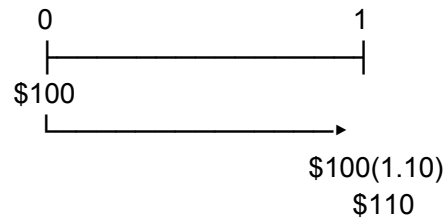
BGN  
PMT=\$6M  
r = 9%  
n = 4  
 $FV_{11} = \$29.91M$

### What is the value today?

1.  $FV_6 = \$19.44M$ ;  $r=9\%$ ;  $n=6$ ;  
 $PV_0 = \$11.59$
2.  $FV_7 = \$21.19M$ ;  $r=9\%$ ;  $n=7$ ;  
 $PV_0 = \$11.59$
3.  $FV_{10} = \$27.44M$ ;  $r=9\%$ ;  $n=10$ ;  
 $PV_0 = \$11.59$
4.  $FV_{11} = \$29.91M$ ;  $r=9\%$ ;  $n=11$ ;  
 $PV_0 = \$11.59$

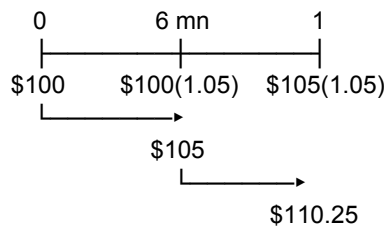
### Compounding Other than Yearly

#### 10% Annual Compounding



### Compounding Other than Yearly

#### 10% Semi-Annual Compounding



### How to Solve?

#### Formula

$$FV_n = PV \times (1 + r/m)^{m \times n}$$

$$FV_1 = \$100 \times (1 + .10/2)^{2 \times 1}$$

$$FV_1 = \underline{\underline{\$110.25}}$$

## How to Solve?

### Calculator

Note:  $r = (r / m)$ ,  $n = (n * m)$

$$\begin{aligned}PV &= 100 \\r &= 10\% / 2 = 5\% \\n &= 1 * 2 = 2\end{aligned}$$

$$FV = \underline{\$110.25}$$

## Problem

If you deposit \$10,000 in this account and earn 11% compounded quarterly, what is the future value in 4½ years?

$$\begin{aligned}PV_0 &= \$10,000 \\r &= 11\% / 4 = 2.75\% \\N &= 4.5 * 4 = 18\end{aligned}$$

$$FV_{4.5} = \$16,295.70$$

## Effective Annual Rates (EAR)

$$EAR = (1 + r/m)^m - 1$$

$r$  = Annual Percentage Rate (APR)

$m$  = number of periods in a year

## Example

If the APR is 10% what is the EAR for these different compounding periods?

annual (m=1)	=	10.00%
semi-annual (m=2)	=	10.25%
quarterly (m=4)	=	10.3813%
monthly (m=12)	=	10.4713%
daily (m=365)	=	10.5156%

## Problem

You see an advertisement in Money magazine from an investment company that offers an account paying a nominal interest rate of 7%.

What is the effective annual rate (EAR) of 7% compounded semi-annually?

What is the EAR of 7% compounded monthly?

## Amortized Loans

Loan that is paid off in equal payments over a set period of time.

Define:

PV = loan amount

$PMT_1 = PMT_2 = \dots = PMT_t$

n = number of payments

r = given

$FV_t = 0$

## Example

Home mortgage: 30 year, \$300,000, 6%, paid monthly

What is the monthly payment?

$PV_0 = \$300,000$ ;  $n = 30 \times 12$ ;  $r = 6\%/12$

**PMT = \$1,798.65**

## Amortization Table

Mn	Beg. Balance	Payment (P&I)	Principal	Interest	End Balance
1	\$300,000.00	\$1,798.65	\$298.65	\$1,500.00	\$299,701.35
2	\$299,701.35	\$1,798.65	\$300.14	\$1,498.51	\$299,401.20
II	II	\$1,798.65	II	II	II
359	\$3,570.50	\$1,798.65	\$1,780.80	\$17.85	\$1,789.70
360	\$1,789.70	\$1,798.65	\$1,789.70	\$8.95	-0-

## 30-year vs. 15-year Mortgage

$PMT_{30} = \$1,798.65$

month 1	I = 1,500.00	P = 298.65
month 2	I = 1,498.51	P = 300.14

$PMT_{15} = \$2,531.57$

month 1	I = 1,500.00	P = 1,031.57
month 2	I = 1,487.34	P = 1,044.23

## Mortgage Balance

30-year vs. 15-year Mortgage

Balance in 5 years  
 $Bal_{30} = \$279,163.07$      $Bal_{15} = \$228,027.30$

Balance in 10 years  
 $Bal_{30} = \$251,057.18$      $Bal_{15} = \$130,946.90$

Balance in 15 years  
 $Bal_{30} = \$213,146.53$      $Bal_{15} = \$0$

## Does this make sense?

Difference in balances in 5 years

$$\$279,163.07 - \$228,027.30 = \$51,135.77$$

Differences in payment

$$\$1,798.6516 - \$2,531.5705 = -\$732.9189$$

$$n = 5, r = 6, PMT = -732.9189, m = 12$$

$$FV = \underline{\underline{\$51,135.77}}$$

## Problem

You purchased your house 6 years ago using a 30-year, \$200,000 mortgage with a contractual rate of 7% that calls for monthly payments.

What is your monthly payment?

What is the loan balance today after making 6 years worth of monthly payments?

How much equity do you have in your house if appreciation has averaged 3% per year?

## Amortization table

5-year, \$100,000 loan, 8% rate, annual payments

Year	Beg Balance	Payment	Principal	Interest	End Balance
1					
2					
3					
4					
5					

## Amortization table

5-year, \$100,000 loan, 8% rate, annual payments

Year	Beg Balance	Payment	Principal	Interest	End Balance
1	\$100,000	25,045.65	17,045.65	8,000.00	82,954.35
2	82,954.35	25,045.65	18,409.30	6,636.35	64,545.06
3	64,545.06	25,045.65	19,882.41	5,163.60	44,663.02
4	44,663.02	25,045.65	21,472.60	3,573.04	23,190.41
5	23,190.41	25,045.65	23,190.41	1,855.23	\$0