UNIVERSITY OF NEVADA LAS VEGAS

Math 181
Recitation Quiz 6 (3.3, 3.5 – 3.7)
SOLUTIONS

Please notify me if you feel there is a typo in need of correction.

1. Differentiate the following, you do not need to simplify

a) $f(x) = \dfrac{\ln x}{1+\ln(2x)}$

SOLN: $f'(x) = \dfrac{[1+\ln(2x)]\cdot(1/x) - \ln x \cdot (1/2x)\cdot 2}{[1+\ln(2x)]^2} = \dfrac{1+\ln(2x)-\ln x}{x[1+\ln(2x)]^2} = \dfrac{1+\ln 2}{x[1+\ln(2x)]^2}$

b) $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

SOLN:

$$y' = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{(e^x+e^{-x})(e^x+e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x+e^{-x})^2}$$

$$= \frac{e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})}{(e^x+e^{-x})^2}$$

$$= \frac{4}{(e^x+e^{-x})^2}$$

c) $y = \tan^{-1}(x - \sqrt{1+x^2})$

SOLN:

$$y' = \frac{1}{1+(x-\sqrt{1+x^2})^2}(x-\sqrt{1+x^2})' = \frac{1}{1+(x-\sqrt{1+x^2})^2}\left(1 - \frac{1}{2}(1+x^2)^{-1/2}(2x)\right)$$

$$= \frac{1}{1+x^2-2x\sqrt{1+x^2}+1+x^2}\left(1-\frac{x}{\sqrt{1+x^2}}\right) = \frac{1}{2+2x^2-2x\sqrt{1+x^2}}\left(\frac{\sqrt{1+x^2}-x}{\sqrt{1+x^2}}\right)$$

$$= \frac{\sqrt{1+x^2}-x}{2\sqrt{1+x^2}\left(1+x^2-x\sqrt{1+x^2}\right)} = \frac{\sqrt{1+x^2}-x}{2\left((1+x^2)\sqrt{1+x^2}-x\sqrt{1+x^2}\sqrt{1+x^2}\right)}$$

$$= \frac{\sqrt{1+x^2}-x}{2\left((1+x^2)\sqrt{1+x^2}-x(1+x^2)\right)} = \frac{\sqrt{1+x^2}-x}{2(1+x^2)\left(\sqrt{1+x^2}-x\right)} = \frac{1}{2(1+x^2)}$$

2. Find the equation for the tangent line to the curve $y = \frac{e^x}{x}$ at the point $x = 1$

SOLN: $y' = \frac{xe^x - e^x}{x^2} \Rightarrow y'(1) = \frac{0}{1} = 0$

$y(1) = e^1$

$y - e = 0(x-1) \Rightarrow y = e$

3. Prove $\frac{d}{dx}(\coth x) = -\text{csch}^2 x$ by either method of your choice:

a) using the definition $\coth x = \frac{\cosh x}{\sinh x}$

SOLN:

$$\begin{aligned}\frac{d}{dx}(\coth x) &= \frac{d}{dx}\left(\frac{\cosh x}{\sinh x}\right) \\ &= \frac{\sinh x \cdot \sinh x - \cosh x \cdot \cosh x}{\sinh^2 x} \\ &= \frac{-(\cosh^2 x - \sinh^2 x)}{\sinh^2 x} \\ &= \frac{-1}{\sinh^2 x} \\ &= -\text{csch}^2 x\end{aligned}$$

b) using the definitions $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$ and $\text{csch}\, x = \frac{2}{e^x - e^{-x}}$.

SOLN:

$$\begin{aligned}\frac{d}{dx}(\coth x) &= \frac{d}{dx}\left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right) \\ &= \frac{(e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})}{(e^x - e^{-x})^2} \\ &= \frac{e^{2x} - 2 + e^{-2x} - (e^{2x} + 2 + e^{-2x})}{(e^x - e^{-x})^2} \\ &= \frac{-4}{(e^x - e^{-x})^2} \\ &= -\left(\frac{2}{e^x - e^{-x}}\right)^2 \\ &= -\text{csch}^2 x\end{aligned}$$

4. Find the numerical value of sech(0) and $\cosh^{-1}(1)$. (3.6 problem 5)

$$\operatorname{sech}(0) = \frac{1}{\cosh(0)} = \frac{2}{e^0 + e^{-0}} = 1$$

$$\begin{aligned} x = \cosh^{-1}(1) \;\Rightarrow\; & \cosh x = 1 \\ & \frac{e^x + e^{-x}}{2} = 1 \\ & e^x + e^{-x} = 2 \\ & e^{2x} - 2e^x + 1 = 0 \\ & e^x = \frac{2 \pm \sqrt{4-4}}{2} = 1 \\ & x = \ln(1) = 0 \end{aligned}$$

5. Prove $\cosh x - \sinh x = e^{-x}$. (3.6 problem 10)

$$\begin{aligned} \cosh x - \sinh x &= \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} \\ &= \frac{(e^x + e^{-x}) - (e^x - e^{-x})}{2} \\ &= \frac{e^{-x} + e^{-x}}{2} \\ &= e^{-x} \end{aligned}$$

6. Find the derivative of $y = e^x \tanh^{-1}(x^3 - 1) + \ln \sqrt[3]{x^2 - 1}$. Specify the domain of *y*. (similar to 3.6 number 38)

$$y = e^x \cdot \tanh^{-1}(x^3 - 1) + \frac{1}{3}\ln(x^2 - 1)$$

$$y' = e^x \cdot \left(\tanh^{-1}(x^3 - 1)\right)' + \tanh^{-1}(x^3 - 1) \cdot e^x + \frac{1}{3}\left(\ln(x^2 - 1)\right)'$$

$$= e^x \cdot \left(\frac{1}{1 - (x^3 - 1)^2} \cdot 3x^2\right) + \tanh^{-1}(x^3 - 1) \cdot e^x + \frac{1}{3}\left(\frac{1}{x^2 - 1}\right) \cdot 2x$$

Domain of $\tanh^{-1} x$ is $-1 < x < 1$.

So $-1 < x^3 - 1 < 1 \Rightarrow 0 < x^3 < 2 \Rightarrow 0 < x < \sqrt[3]{2}$ or $x \in (0, \sqrt[3]{2})$.

Domain of $\ln x$ is $x > 0$.

So $x^2 - 1 > 0 \Rightarrow (x-1)(x+1) > 0 \Rightarrow (-\infty, -1) \cup (1, \infty)$.

So total, the domain is $(1, \sqrt[3]{2})$.

7. Find the limit of $\lim_{x\to 0}\frac{\cos mx-\cos nx}{x^2}$ using L'Hopital's rule. (3.7 number 16)

$$\lim_{x\to 0}\frac{\cos mx-\cos nx}{x^2}\underset{LH}{=}\lim_{x\to 0}\frac{-m\sin mx+n\sin nx}{2x}$$
$$\underset{LH}{=}\lim_{x\to 0}\frac{-m^2\cos mx+n^2\cos nx}{2}$$
$$=\frac{-m^2+n^2}{2}$$

8. Find the limit of $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$ using L'Hopital's rule. (3.7 number 30)

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to 1}\left(\frac{x-1-\ln x}{(x-1)\ln x}\right)$$
$$=\lim_{x\to 1}\left(\frac{x-1-\ln x}{x\ln x-\ln x}\right)\underset{LH}{=}\lim_{x\to 1}\left(\frac{1-\frac{1}{x}}{x\cdot\frac{1}{x}+\ln x-\frac{1}{x}}\right)$$
$$=\lim_{x\to 1}\left(\frac{x-1}{x-1+x\ln x}\right)\underset{LH}{=}\lim_{x\to 1}\left(\frac{1}{1+x\cdot\frac{1}{x}+\ln x}\right)$$
$$=\lim_{x\to 1}\left(\frac{1}{2+\ln x}\right)=\frac{1}{2}$$

9. Find the limit of $\lim_{x\to 0^+}\sin x\ln x$ using L'Hopital's rule. (3.7 number 24)

$$\lim_{x\to 0^+}\sin x\ln x=\lim_{x\to 0^+}\frac{\ln x}{\csc x}$$
$$\underset{LH}{=}\lim_{x\to 0^+}\frac{1/x}{-\csc x\cdot\cot x}=\lim_{x\to 0^+}\frac{-\sin x}{x\cot x}$$
$$=\lim_{x\to 0^+}\frac{-\sin^2 x}{x\cos x}\underset{LH}{=}\lim_{x\to 0^+}\frac{-2\sin x\cos x}{x(-\sin x)+\cos x}$$
$$=\frac{-2(0)(1)}{(0)(-0)+1}=0$$

10. Find the limit of $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$ using L'Hopital's rule. (3.7 number 34)

$$\ln\left(1+\frac{a}{x}\right)^{bx} = bx\cdot\ln\left(1+\frac{a}{x}\right)$$

$$\lim_{x\to\infty}\left[bx\cdot\ln\left(1+ax^{-1}\right)\right] = b\cdot\lim_{x\to\infty}\frac{\ln\left(1+ax^{-1}\right)}{x^{-1}}$$

$$\underset{LH}{=} b\cdot\lim_{x\to\infty}\frac{\frac{1}{1+ax^{-1}}\cdot(-ax^{-2})}{-x^{-2}} = ab\cdot\lim_{x\to\infty}\frac{1}{1+\frac{a}{x}} = ab$$

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = \lim_{x\to\infty}\exp\left[\ln\left(1+\frac{a}{x}\right)^{bx}\right] = \exp\lim_{x\to\infty}\ln\left(1+\frac{a}{x}\right)^{bx} = \exp(ab)$$