Section 4.4 –Curve Sketching

A Quick Guideline:

- When looking to sketch a function, it is a good idea to start with the function type.
- Find the domain.
- Find any asymptotes.
- Find any limits (if needed).
- Find the first and second derivative.
- Find any critical points, inflection points.
- Determine when it is increasing/decreasing, concave up/down.
- See the **supplemental** to sketching for a more detailed guideline.

- *Example. Sketch* $y = \dfrac{x}{(x-1)^2}$

This is a rational expression. We expect a restricted domain and vertical/horizontal asymptotes.

Domain:

$(x-1)^2 = 0$ when $x = 1$

Since x = 1 does not make the top go to zero, it is a vertical asymptote.

Roots:

x = 0 is a root. Note $(x-1)^2\big|_0 \neq 0$

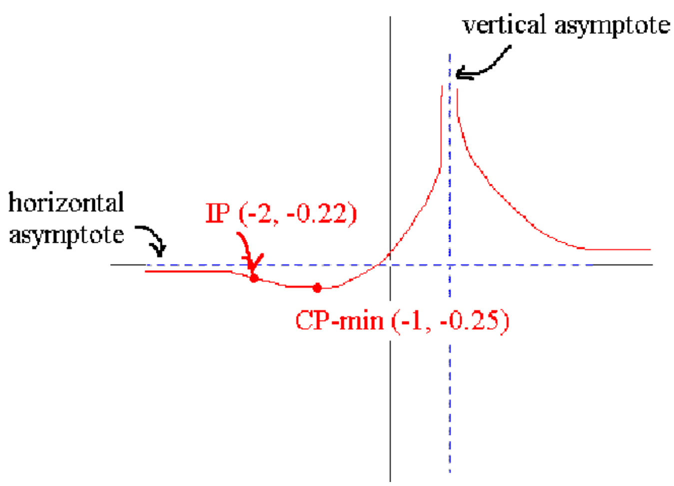
Horizontal Asymptote:

$\lim\limits_{x\to\infty} \dfrac{x}{(x-1)^2}$ as x gets large, it behaves like $\dfrac{x}{x^2} = \dfrac{1}{x}$. So there is a H.A. at 0.

Evaluating the first derivative:

$$y' = \frac{(x-1)^2 - 2x(x-1)^1}{(x-1)^4}$$
$$= \frac{-1-x}{(x-1)^3}$$
$$= -\frac{x+1}{(x-1)^3}$$

$y' = 0$ when $x = -1$, undefined when x = 1.

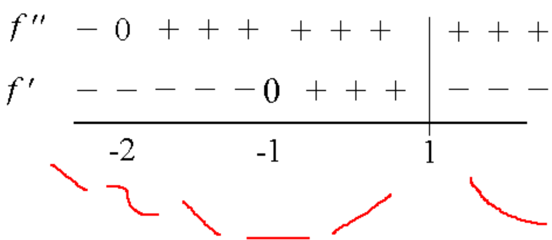


Evaluating the second derivative:

$$y'' = \frac{-(x-1)^3 - (-x-1)3(x-1)^2}{(x-1)^6}$$
$$= \frac{2x+4}{(x-1)^4}$$

There is an inflection point at x = –2

So doing a sign diagram we see the basic shape



Evaluating $y(-1) = \dfrac{-1}{(-1-1)^2} = -0.25$ and $y(-2) = \dfrac{-2}{(-2-1)^2} = -0.22$

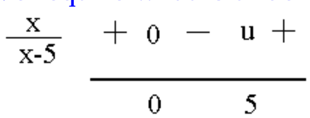
We can also evaluate the limit as x tends to 1 from the right and left to verify.

- *Example. Sketch* $y=\sqrt{\dfrac{x}{x-5}}$.

This is a composite of a root and a rational expression.

Domain:

We require what is under the radical to be positive.



We restrict our domain to $x \le 0,\ x > 5$.

Asymptotes:

There will be a vertical asymptote at $x = 5$, and $\lim_{x\to 5^+} y = +\infty$.

There will be a horizontal asymptote at 1.

Also note that $\lim_{x\to 0^-} y = 0$.

First derivative:

$$y' = \frac{1}{2}\frac{1}{\sqrt{\dfrac{x}{x-5}}}\frac{x-5-x}{(x-5)^2}.$$

$$= \frac{-5}{2}\frac{1}{\sqrt{x(x-5)^3}}$$

There are critical points at $x = 5$ and $x = 0$, but no horizontal tangents.
Notice that the derivative is always negative.

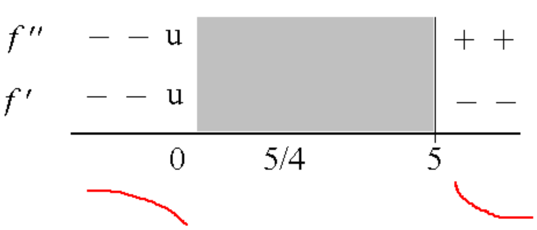
Second derivative:

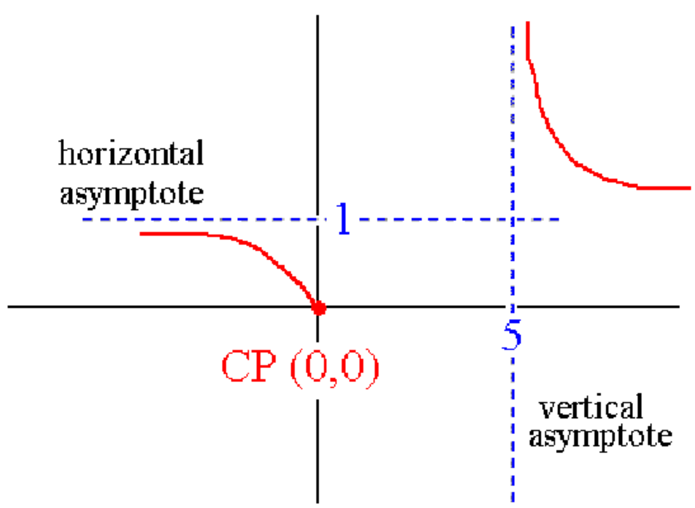
$$y'' = \frac{-5}{2}\left(x^{-1/2}\cdot\frac{-3}{2}(x-5)^{-5/2} + (x-5)^{-3/2}\cdot\frac{-1}{2}x^{-3/2}\right)$$

$$= \frac{-5}{4}\frac{5-4x}{\sqrt{x^3(x-5)^5}}$$

There could be an inflection point at $5-4x=0,\ x=\dfrac{5}{4}$ but it is not in our domain.

Sign diagram and graph:





- *Example. Sketch* $y = x(\ln x)^2$

Domain:

The function is only defined for $x > 0$

Limit:

Let's look at the limit as x tends to zero.

$$\lim_{x\to 0} x(\ln x)^2 = \lim_{x\to 0} \frac{(\ln x)^2}{x^{-1}}$$

$$= \lim_{x\to 0} \frac{2(\ln x)\ x^{-1}}{-x^{-2}}$$

$$= \lim_{x\to 0} \frac{-2(\ln x)}{x^{-1}} \quad .$$

$$= -2\lim_{x\to 0} \frac{1/x}{-1x^{-2}}$$

$$= -2\lim_{x\to 0}(-x)$$

$$= 0$$

First derivative:

$$y' = 2(\ln x) + (\ln x)^2$$

$$= (2 + \ln x)(\ln x)$$

$y' = 0$ when

$2 + \ln x = 0$ or $\ln x = 0$

$x = e^{-2} \approx 0.14$ $\quad x = 1$

Second derivative:

$$y'' = (2 + \ln x)\frac{1}{x} + \frac{1}{x}\ln x$$

$$= \frac{1}{x}(2 + 2\ln x)$$

$y'' = 0$ when

$2 + 2\ln x = 0$

$x = e^{-1} \approx 0.37$

Sign diagram and rough sketch: