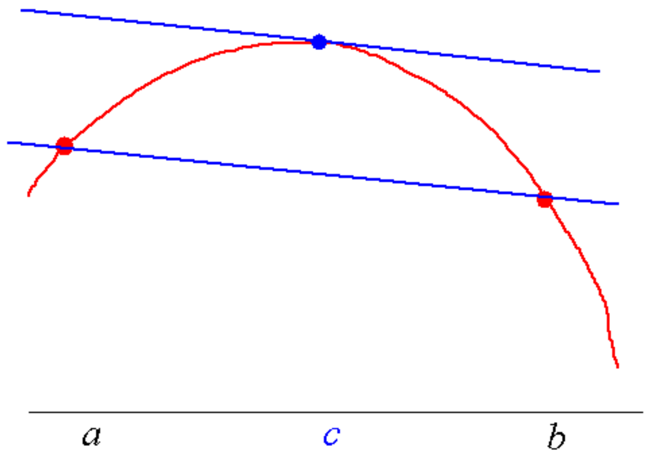
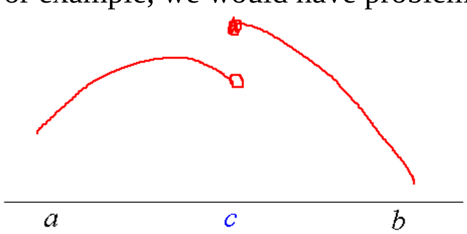
Section 4.2 – The Mean Value Theorem

Recall:

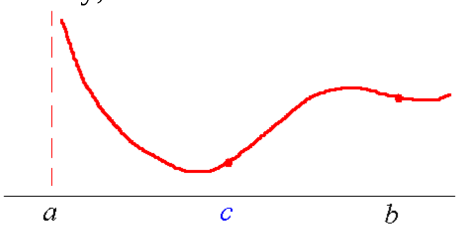
- When we first looked at the definition of the derivative, we started by looking at the slope of secant lines. Originally we picked the point *c* and another (which we called *c* + *h*).
- We could have also taken any two points surrounding *c*, and looked at the line connecting *a* and *b*. And then we could have moved this up until it was tangent to the curve at the point *c*.



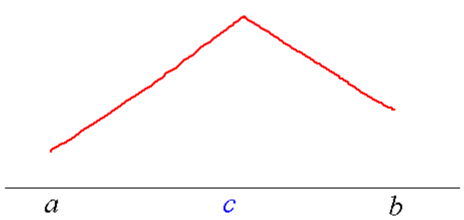
- Q: What conditions were important to us at that time?
  A: The function needed to be continuous on [*a*,*b*].
- For example, we would have problems if we had a function that looked like:



Actually, it should be continuous at *a* and *b* as well. *f*(*a*) not defined below causes problems:



It should be differentiable on the interval (*a*,*b*), or we could run into a problem like:



where the derivative at *c* is not even defined.

- So to be able to do this in a general sense, we need to specify that f is **continuous on [*a*,*b*] and differentiable on (*a*,*b*)**.
- Q: What would be the slope of this secant line if $f(a) = f(b)$?
  A: Zero.

The Theorems:

- **Mean Value Theorem:** Let f be a function that is continuous on $[a,b]$ and differentiable on (a,b). Then there is a c in (a,b) with $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.
- **Rolle's Theorem:** Let f be a function that is continuous on $[a,b]$ and differentiable on (a,b) with $f(a) = f(b)$. Then there is a c in (a,b) with $f'(c) = 0$.

Examples:

- *Example. Verify* $f(x) = x\sqrt{x+6},\ [-6,0]$ *satisfies Rolles and find all such c's.*

  $f(x)$ is continuous on [–6,0] and is differentiable on (–6,0). Note we are not *proving* only verifying.

  $f(-6) = (-6)\sqrt{-6+6} = 0$

  $f(0) = (0)\sqrt{0+6} = 0$

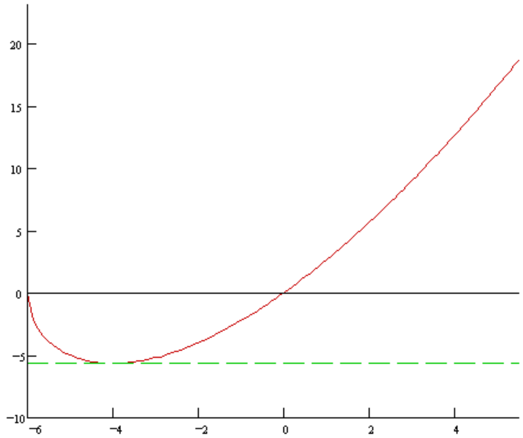
  So it satisfies Rolles.

  Finding the derivative we have:

  $$f'(x) = \frac{x}{2\sqrt{x+6}} + \sqrt{x+6}$$
  $$= \frac{3(x+4)}{2\sqrt{x+6}}$$

  Setting this equal to zero and solving for x we have:
  $f'(x) = 0$ when $x = -4$

  Note that this is in (–6,0).



- *Example. Let* $f(x)=(x-1)^{-2}$. *Show that f(0)=f(2) but there is no number c in (0,2) so* $f'(c)=0$. *Why not?*

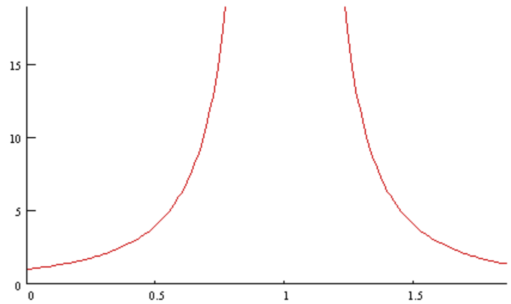
First of all it is important to note that *f* has a vertical asymptote at *x* = 1, which is in [0,2].

Therefore, *f* is not continuous on [0,2], which is a requirement for Rolles.
This is enough to answer the question, but let's look further.

$f(0)=(0-1)^{-2}=1$, $f(2)=(2-1)^{-2}=1$, so *f*(0) = *f*(2).

$$f'(x)=-2(x-1)^{-3}$$
$$=\frac{-2}{(x-1)^3}$$

So the derivative is never zero. We can verify this with the graph.



- *Example. Verify that* $f(x)=\dfrac{x}{x+2}$ *satisfies the MVT on [1,4]. Find c.*

  f is continuous on [1,4] (there is an asymptote, but it is at $x = -2$).

  f is differentiable on (1,4), and

$$f'(x)=\frac{(x+2)-x(1)}{(x+2)^2}$$
$$=\frac{2}{(x+2)^2}$$

$$f(1)=\frac{1}{1+2}=\frac{1}{3}$$

$$f(4)=\frac{4}{4+2}=\frac{4}{6}=\frac{2}{3}$$

So $$\frac{f(b)-f(a)}{b-a}=\frac{2/3-1/3}{4-1}$$
$$=\frac{1/3}{3}$$
$$=\frac{1}{9}$$

Setting the derivate equal to this value, we find:

$$\frac{2}{(x+2)^2}=\frac{1}{9}$$
$$(x+2)^2=18$$
$$x=\pm\sqrt{18}-2=2.24,\ -6.24$$

On the interval given, we find $c = 2.24$