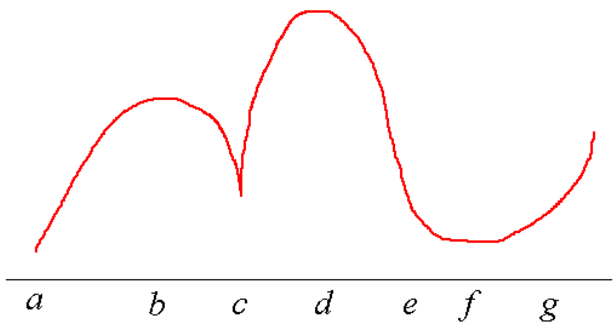
Section 4.1 – Max and Min Values

Horizontal Tangents:

- We have looked at graphs and identified **horizontal tangents**, or places where the slope of the tangent line is zero.
- Q: For which x values does the following function have a horizontal tangent?



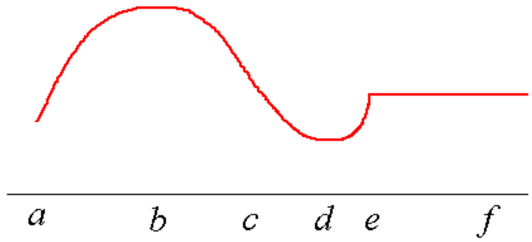
A: There is a horizontal tangent at *b, d* and *f*. Note that *c* is not a horizontal tangent.

- Take a look at each horizontal tangent again. Notice that to the left and to the right, the sign of the derivative is different.
- For example, to the left of point *b* the derivative is **positive** (function is **increasing**) and to the right of point *b* the derivative is **negative** (function is **decreasing**).
- Also notice that when we say "to the left of *b*" and "to the right of *b*" we mean values that are really "close" to the value of *b*. Another point, say *a* or *c*, would be considered too "far" away.
- Also note that the point itself is a maximum or minimum compared to the values *near* it. This is called a **relative max** or **relative min** (sometimes called **local max** or **local min**).
- Notice point *c*. It is not a horizontal tangent, but to the left of *c* the function is decreasing and to the right of *c* the function is increasing. It is also considered to be a relative min, but the derivative there is undefined (recall the derivative is undefined at sharp corners).

Critical Points:

- A **critical point** (sometimes called a **critical number**) of a function $f(x)$ is when $f'(x) = 0$ or $f'(x)$ is undefined.
- Q: What are the critical points of the graph pictured above?
  A: *b, c, d* and *f*.
- Critical points include both horizontal tangents (derivative is zero), and places where the derivative is undefined (which could be corners, jumps or asymptotes). This does not mean the point is a maximum or minimum value. But if a point is a max or min, it will be a critical point.

- *Example. Identify the critical points, horizontal tangents, local max/min in the picture below.*



Critical points: *b, d, e,* points between *e* and *f*
Horizontal tangents: *b, d,* and *e* through *f*
Relative Max: *b*
Relative Min: *d*
Notice that *e* could be considered a max, but it for sure considered a critical point.

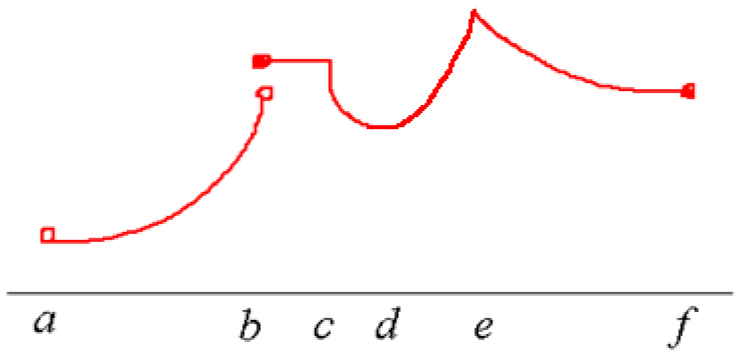
Also, the points between *e* and *f* are critical points (because the derivative is zero there) but they are not max or min values.

- So to recap -- relative max or min implies critical point, but critical point does not imply relative maximum or minimum.

Formal Definitions:

- A function f has a **local maximum** at c if $f(c) \geq f(x)$ for all x near c.
- A function f has a **local minimum** at c if $f(c) \leq f(x)$ for all x near c.
- Note that the equality allowed in the definition (according to our book) is what makes *e* in our example above a maximum value. Some people/books will not allow for equality in the definitions above, and use strict inequalities. For purposes of this class, we will stick with the books definition.
- A function f has an **absolute max** (or **global max**) at c if $f(c) \geq f(x)$ for all x in the domain of f.
- A function f has an **absolute min** (or **global min**) at c if $f(c) \leq f(x)$ for all x in the domain of f.
- Again, some people/books would use strict inequalities.
- Q: What is the difference between a global max and a local max?
  A: A global max will be larger than all values in the domain, and a local max will only be larger than the values around it.
- Q: Is it possible to have more than one global max (or min)?
  A: Yes! Because we allow for equality, it is possible that we can have more than one global maximum (or min).

- *Example: For each point in the picture below, is it a… local max, local min, global max, global min, critical point?*



  a is a local min (it is not a global min, even though the limit is less than all other values, because it does not attain the value at *a*). *b* is a critical point (the derivative is not defined there), and a local max (because we allow for equality in the definition). *c* is a critical point (derivative not defined) and a local max (again, because we allow for equality). *d* is a local min. *e* is a critical point, a local max, and a global max. *f* is a local min (this is going by the definition in our book, which allows only for values in the domain), you would be just as correct classifying it as nothing, because the graph is cut off.

- **Fermat's Theorem** says that if f has a local max or min at c, and if $f'(c)$ exists, then $f'(c) = 0$.

How to Find and Identify Critical Points for an Equation:

- To find critical points from an equation, simply find the derivative. Anywhere the derivative is zero or undefined you have a critical point.
- If the critical point, c, comes from when the derivative is zero ($f'(c) = 0$), it could be…
  – A local max
  $f''(c) < 0$. Also, $f'(c^-) > 0$ and $f'(c^+) < 0$ (c^- is a value immediately to the left of c, c^+ is right).
  – A local min
  $f''(c) > 0$. Also, $f'(c^-) < 0$ and $f'(c^+) > 0$.
  – A global max on $[a, b]$
  $f''(c) < 0$, f is continuous on the interval $[a,b]$ and $f(c) \geq f(d) \quad \forall d \in [a,b]$.
  – A global min on $[a, b]$
  $f''(c) > 0$, f is continuous on the interval $[a,b]$ and $f(c) \leq f(d) \quad \forall d \in [a,b]$.
  – An inflection point
  $f''(c) = 0$.
- If the critical point comes from when the derivative is undefined, it could be…
  – A "sharp" corner.
  – A jump discontinuity.
  – An infinite discontinuity (asymptote).
  All of these will be identifiable in the original function.

- *Example. Find the cp's (and identify) for* $f(x)=\dfrac{x^2-4}{x^2+4}$ *on* [–4,4]

$$f'(x)=\frac{(x^2+4)2x-(x^2-4)2x}{(x^2+4)^2}$$
$$=\frac{16x}{(x^2+4)^2}$$

Notice the original function is defined for all x, but the domain is restricted to [–4,4].
$f'(x)$ is never undefined. $f'(x)=0$ when $x = 0$. This is the only critical point.

Let's classify this critical point using the second derivative test:

$$f''(x)=\frac{16(x^2+4)^2-64x^2(x^2+4)}{(x^2+4)^4}$$
$$=\frac{16[x^2+4-4x^2]}{(x^2+4)^3}$$
$$=\frac{16[4-3x^2]}{(x^2+4)^3}$$

$f''(0)>0$, so $x = 0$ is a relative min.

Is it a global min on [–4,4]?

There is only one critical point, so to test for global max, we need only test the value we found and the endpoints.

$$f(-4)=\frac{(-4)^2-4}{(-4)^2+4}=0.6,$$
$$f(4)=\frac{4^2-4}{4^2+4}=0.6,$$
$$f(0)=\frac{0-4}{0+4}=-1$$

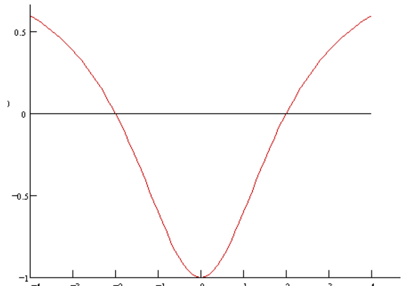
Yes, it is a global min.

Is there a global max?

Yes. Recall that $f(-4)=f(4)=0.6$

This means that f has a global max at +4 and –4.

- Q: Why could there not be a value that exceeds 0.6 on [–4,4]?
  A: Because in that range, there is only one critical point and no undefined values. This means there is nothing 'special' going on other than the local/global min we found already. In fact, the picture of the function is below.



- *Example. Find and identify any critical points for* $f(x)=e^{-x}-e^{-2x}$ *on* [0,1].

  $f'(x)=-e^{-x}+2e^{-2x}$

  Notice the original function is defined for all *x*, but the domain is restricted to [0,1].

  $f'(x)$ is never undefined. $f'(x)=0$ when

  $-e^{-x}+2e^{-2x}=0$

  $2e^{-2x}=e^{-x}$

  $2=e^{x}$

  $x=\ln 2$

  This is the only critical point.

  Let's classify this critical point using the second derivative test:

  $f''(x)=e^{-x}-4e^{-2x}$

  $f''(\ln 2)<0$, so $x = 0$ is a relative max.

  Is ln(2) a global max on [0,1]?

  We need to test the endpoints and any critical points, recall the function is defined in the domain.

  $f(0)=e^{0}-e^{0}=0$

  $f(1)=e^{-1}-e^{-2}=0.23$

  $f(\ln 2)=e^{-\ln 2}-e^{-2\ln 2}=0.25$

  Yes, it is a global max.

  Is there a global min?

  f has a global min at 0 since there are no other critical points and *f*(0) is less than the rest.

  In fact, the function looks like