Section 3.7 – Indeterminate Forms and L'Hopital's Rule

Recall Limits:

- We were working with limits in Chapters 1 and 2. Recall when we encountered a 0/0 in a rational expression, we could perhaps "fix" the behavior and analyze the limit by factoring and canceling terms.

- *Example. Find* $\lim_{x\to 1}\frac{x^2-1}{x-1}$

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}$$
$$=\lim_{x\to 1}\frac{(x+1)}{1}=2$$

- Now we have an 'easier' way… If we are taking the limit and run across 0/0, we can take the derivative of the top and bottom and try again.

- *Example above (reworked)*

$$\lim_{x\to 1}\frac{x^2-1}{x-1}\overset{LH}{=}\lim_{x\to 1}\frac{2x}{1}=2$$

Note that at the first step we have 0/0.

Differentiating the top $(x^2-1)'=2x$ and differentiating the bottom, $(x-1)'=1$

We try again and get 2.

- This is what is known as applying L'Hopital's Rule.

Indeterminate Forms:

- The following forms are 'indeterminate' meaning we are not sure what happens and need to investigate further:

$$\frac{0}{0}\quad \frac{\infty}{\infty}\quad 0\cdot\infty\quad \infty-\infty\quad 0^0\quad \infty^0\quad 1^\infty$$

- L'Hopitals Rule applies when you have either of the indeterminate forms $\left(\frac{0}{0}\text{ or }\frac{\infty}{\infty}\right)$.

- You can use L'Hopitals Rule more than once so long as you still have the indeterminate form above.

- *Example. Evaluate* $\lim_{x\to -2}\frac{x+2}{x^2+3x+2}$

First note that $x+2\big|_{x=-2}=0$ and $x^2+3x+2\big|_{x=-2}=0$

So we can use L'Hopitals Rule

$$\lim_{x\to -2}\frac{x+2}{x^2+3x+2}\overset{LH}{=}\lim_{x\to -2}\frac{1}{2x+3}$$
$$=\frac{1}{-1}=-1$$

- *Example. Evaluate* $\lim\limits_{x\to\pi/2}\dfrac{1-\sin x}{\csc x}$

  First note that

  $$1-\sin x\Big|_{x=\pi/2}=1-1=0$$

  $$\csc x\Big|_{x=\pi/2}=\frac{1}{\sin x}\Bigg|_{x=\pi/2}=1$$

  So we can evaluate this directly (without L'Hopital)

  $$\lim_{x\to\pi/2}\frac{1-\sin x}{\csc x}=\frac{0}{1}=0.$$

- *Example. Evaluate* $\lim\limits_{x\to-\infty}x^2e^x$.

  $\lim\limits_{x\to-\infty}x^2=\infty$ and $\lim\limits_{x\to-\infty}e^x=0$

  So we have the indeterminate form $0\cdot\infty$.

  We rewrite the problem so we have the form we need (0/0)… $\lim\limits_{x\to-\infty}\dfrac{x^2}{e^{-x}}$.

  Then using L'Hopitals Rule (twice) we have

  $$\begin{aligned}\lim_{x\to-\infty}\frac{x^2}{e^{-x}}&\overset{LH}{=}\lim_{x\to-\infty}\frac{2x}{-e^{-x}}\\&\overset{LH}{=}\lim_{x\to-\infty}\frac{2}{e^{-x}}\\&=\lim_{x\to-\infty}2e^x\\&=0\end{aligned}$$

- *Example. Evaluate* $\lim\limits_{x\to0}(\csc x-\cot x)$

  $\lim\limits_{x\to0}\csc x=\lim\limits_{x\to0}\dfrac{1}{\sin x}=\infty$ and $\lim\limits_{x\to0}\cot x=\lim\limits_{x\to0}\dfrac{1}{\tan x}=\infty$.

  So we have the indeterminate form $\infty-\infty$.

  We then rewrite the problem so we have the form we need

  $$\lim_{x\to0}\left(\frac{1}{\sin x}-\frac{\cos x}{\sin x}\right)=\lim_{x\to0}\left(\frac{1-\cos x}{\sin x}\right)$$

  Note now that we have $(1-\cos x)\Big|_{x=0}=1-1=0$ and $\sin x\Big|_{x=0}=0$. Which is the form we need.

  Then using L'Hopitals Rule we have

  $$\begin{aligned}\lim_{x\to0}\left(\frac{1-\cos x}{\sin x}\right)&\overset{LH}{=}\lim_{x\to0}\left(\frac{\sin x}{\cos x}\right)\\&=\frac{0}{1}\\&=0\end{aligned}$$

- *Example. Evaluate* $\lim\limits_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1}$

Notice that $\lim\limits_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)=1$ and $\lim\limits_{x\to\infty} 2x+1=\infty$.

So we have the indeterminate form 1^{∞}.

Note that

$$\ln\left(\frac{2x-3}{2x+5}\right)^{2x+1}=(2x+1)\cdot\ln\left(\frac{2x-3}{2x+5}\right) \text{ and}$$

$$\lim_{x\to\infty}\frac{1}{2x+1}=0 \text{ and } \lim_{x\to\infty}\ln\left(\frac{2x-3}{2x+5}\right)=0.$$

Using L'Hopitals Rule we find

$$\begin{aligned}
\lim_{x\to\infty}\frac{\ln\left(\frac{2x-3}{2x+5}\right)}{(2x+1)^{-1}} &\overset{LH}{=} \lim_{x\to\infty}\frac{\left(\frac{2x+5}{2x-3}\right)\left(\frac{(2x+5)2-(2x-3)2}{(2x+5)^2}\right)}{-2(2x+1)^{-2}}\\
&=\lim_{x\to\infty}\frac{\frac{1}{2x-3}\frac{16}{2x+5}}{-2}(2x+1)^2\\
&=\lim_{x\to\infty}\frac{-8(2x+1)^2}{(2x-3)(2x+5)}\\
&=-8
\end{aligned}$$

So we have

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1} &= \exp\left(\ln\ \lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1}\right)\\
&=\exp\left(\lim_{x\to\infty}\ln\left(\frac{2x-3}{2x+5}\right)^{2x+1}\right)\\
&=\exp\left(\lim_{x\to\infty}(2x+1)\ln\left(\frac{2x-3}{2x+5}\right)\right)\\
&=e^{-8}
\end{aligned}$$