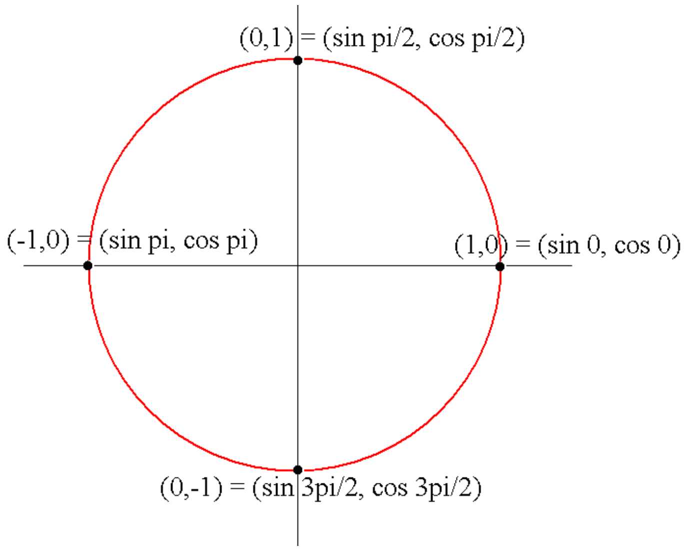
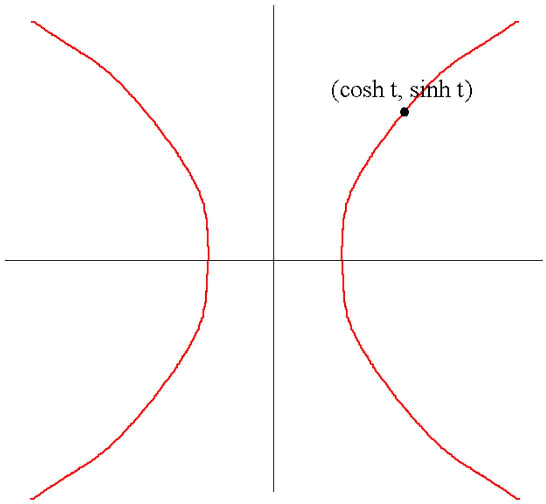
Section 3.6 – Hyperbolic Functions

Definitions:

- Recall the trig functions are related to the unit circle $x^2 + y^2 = 1\ldots$



- In a similar way, hyperbolic functions are related to the hyperbolic function $x^2 - y^2 = 1\ldots$



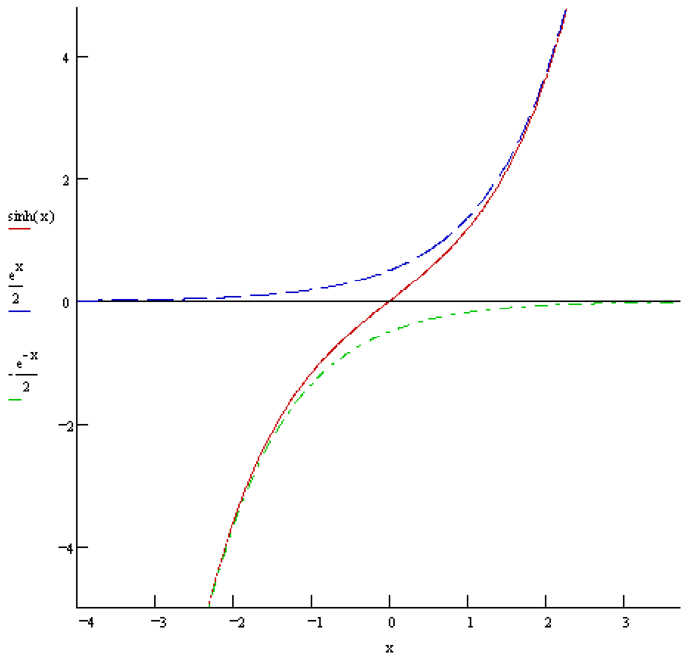
- They can be expressed in terms of linear combinations of exponential growth and decay.
- $\sinh x = \dfrac{e^x - e^{-x}}{2}$

  $\cosh x = \dfrac{e^x + e^{-x}}{2}$
- For the regular trig functions, sine and cosine give rise to tangent, cotangent, secant and cosecant.
- In a similar way, hyperbolic sine and hyperbolic cosine give rise to…

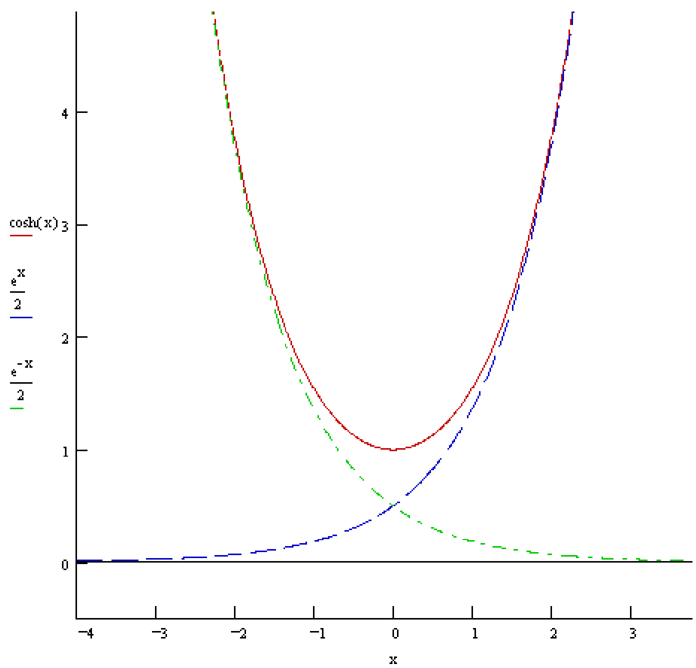
  $\tanh x = \dfrac{\sinh x}{\cosh x}$ $\quad$ $\text{csch} x = \dfrac{1}{\sinh x}$ $\quad$ $\text{sech} x = \dfrac{1}{\cosh x}$ $\quad$ $\coth x = \dfrac{\cosh x}{\sinh x}$

<u>Graphs of Hyperbolic Sine and Cosine:</u>

- $\sinh x = \dfrac{e^x - e^{-x}}{2}$



- $\cosh x = \dfrac{e^x + e^{-x}}{2}$



Properties of Hyperbolic Functions:

- $\sinh(-x) = \dfrac{e^{-x} - e^{-(-x)}}{2} = -\dfrac{e^{x} - e^{-x}}{2} = -\sinh x$
- $\cosh(-x) = \dfrac{e^{-x} + e^{-(-x)}}{2} = \dfrac{e^{x} + e^{-x}}{2} = \cosh x$
- Q: From our definitions of **even** and **odd**, how can we classify sinh and cosh?
  A: Sinh is an odd function, cosh is an even function.
- Q: What can we then say about their symmetry?
  A: Sinh is symmetric about the origin, cosh is symmetric about the *y* axis.
- $\cosh^2 x - \sinh^2 x = 1$
- And the last two properties are:
  $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
  $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

Derivatives:

- $\dfrac{d}{dx}\sinh x = \dfrac{d}{dx}\left(\dfrac{e^{x} - e^{-x}}{2}\right) = \dfrac{e^{x} + e^{-x}}{2} = \cosh x$.
  The derivative of sinh is cosh
- $\dfrac{d}{dx}\cosh x = \dfrac{d}{dx}\left(\dfrac{e^{x} + e^{-x}}{2}\right) = \dfrac{e^{x} - e^{-x}}{2} = \sinh x$.
  The derivative of cosh is sinh
- $\dfrac{d}{dx}\tanh x = \dfrac{d}{dx}\left(\dfrac{\sinh x}{\cosh x}\right) = \dfrac{\cosh x(\cosh x) - \sinh x(\sinh x)}{\cosh^2 x} = \dfrac{1}{\cosh^2 x} = \text{sech}^2 x$
- Q: Use your understanding of the chain rule and the property that csch*x* = 1/sinh*x* to find $\dfrac{d}{dx}\text{csch}x$.

  A: $\dfrac{d}{dx}\text{csch}x = \dfrac{d}{dx}\dfrac{1}{\sinh x}$
  $= (-1)(\sinh x)^{-2} \cosh x$
  $= \dfrac{-\cosh x}{\sinh^2 x}$
  $= -\coth x \cdot \text{csch}x$

- Q: Use your understanding of the property $\operatorname{csch} x = \dfrac{2}{e^x - e^{-x}}$ to find $\dfrac{d}{dx}\operatorname{csch} x$.

  A: $$\begin{aligned}\frac{d}{dx}\operatorname{csch} x &= \frac{d}{dx}\frac{2}{e^x - e^{-x}} \\ &= 2(-1)(e^x - e^{-x})^{-2}(e^x + e^{-x}) \\ &= -\frac{e^x + e^{-x}}{e^x - e^{-x}}\frac{2}{e^x - e^{-x}} \\ &= -\coth x \cdot \operatorname{csch} x\end{aligned}.$$

- Q: Use your understanding of the chain rule and the property that sech$x$ = 1/cosh$x$ to find $\dfrac{d}{dx}\operatorname{sech} x$.

  A: $$\begin{aligned}\frac{d}{dx}\operatorname{sech} x &= \frac{d}{dx}\frac{1}{\cosh x} \\ &= (-1)(\cosh x)^{-2}(\sinh x) \\ &= \frac{-\sinh x}{\cosh^2 x} \\ &= -\tanh x \cdot \operatorname{sech} x\end{aligned}.$$

- Q: Use your understanding of the chain rule and the property that coth$x$ = cosh$x$/sinh$x$ to find $\dfrac{d}{dx}\coth x$.

  A: $$\begin{aligned}\frac{d}{dx}\coth x &= \frac{d}{dx}\frac{\cosh x}{\sinh x} \\ &= \frac{\sinh x(\sinh x) - \cosh x(\cosh x)}{\sinh^2 x} \\ &= \frac{-1}{\sinh^2 x} \\ &= -\operatorname{csch}^2 x\end{aligned}$$

Inverse Hyperbolic Functions:

- There are functions that ‘undo’ the hyperbolic functions

  $y = \sinh^{-1} x \Leftrightarrow \sinh y = x$

  $y = \cosh^{-1} x \Leftrightarrow \cosh y = x \quad y \geq 0$

  $y = \tanh^{-1} x \Leftrightarrow \tanh y = x$

- Because hyperbolic functions are related to exponentials, their inverses are related to logs

  $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}) \quad x \in R$

  $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \quad x \geq 1$

  $\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right) \qquad -1 < x < 1$

- Let's try to see why the first equation works…

  We want to find the inverse of the function $y = \sinh x$

  As we usually proceed to find an inverse, we swap *x* and *y* and solve for *y*

  $x = \sinh y = \dfrac{e^y - e^{-y}}{2}$

  $2x = e^y - e^{-y}$

  $2xe^y = e^{2y} - 1$

  $(e^y)^2 - 2x(e^y) - 1 = 0$

  $e^y = \dfrac{2x \pm \sqrt{4x^2 - 4(1)(-1)}}{2(1)} = x \pm \sqrt{x^2 + 1}$ by the quadratic formula.

  So either $e^y = x - \sqrt{x^2 + 1}$. But $x - \sqrt{x^2 + 1} < 0$ for all *x*. This is not valid.

  or $e^y = x + \sqrt{x^2 + 1} \Rightarrow y = \ln(x + \sqrt{x^2 + 1})$

- We can also differentiate any of the inverse trig functions, these formulas are in your book.