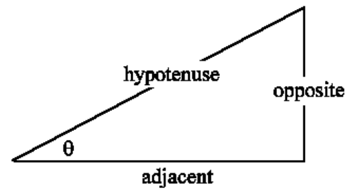
Section 3.5 – Inverse Trig Functions

Recall the Trigonometric Functions:

- The six trig functions are defined in terms of right triangles (see figure).
- The two "most important" being sine and cosine.

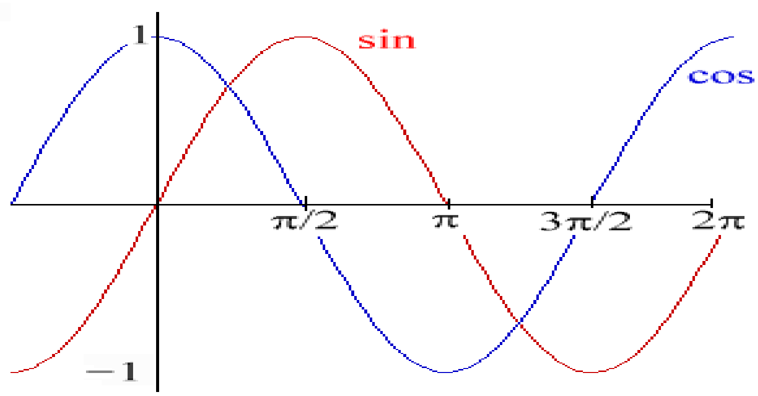
$$\sin\theta = \frac{\text{opp.}}{\text{hyp.}}$$
$$\cos\theta = \frac{\text{adj.}}{\text{hyp.}}$$



- All other trig functions can be defined in terms of sine and cosine, so remember the definitions above, and the relationships between them and the others…

$$\sec\theta = \frac{1}{\cos\theta} \quad \tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\csc\theta = \frac{1}{\sin\theta} \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

- Evaluating trig functions boils down to some fundamentals.

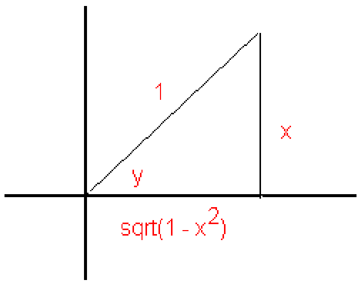


- First, remember the basic shape of sin(x) and cos(x), along with when they are 0, and ±1. From the figure you can see they are periodic, both with a period of 2π.

The Inverse Trig Functions:

- $\sin^{-1} x = y$ *iff* $\sin y = x$ and $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$
- $\cos^{-1} x = y$ *iff* $\cos y = x$ and $0 \le y \le \pi$
- $\tan^{-1} x = y$ *iff* $\tan y = x$ and $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$
- Note the domain restriction for the inverse trig functions.

- To be able to determine the derivative of sine inverse, we first must recall some trig… If we want to graph $\sin y = x$ we would have



This would mean that $\cos(y) = \sqrt{1-x^2}$

Using implicit differentiation, we find

$$\frac{d}{dx}(\sin y = x)$$
$$y' \cos y = 1$$
$$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

<u>The Inverse Trig Functions:</u>

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- $\tan^{-1} x = y$ *iff* $\tan y = x$ and $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$
- Note the domain restriction for the inverse trig functions.
- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ $\qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$

  $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$ $\qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

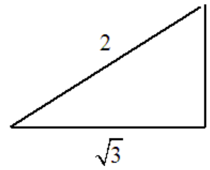
  $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ $\qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

Some Example Problems:

- *Example. Find the value of* $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ *and use it to solve* $\cos x = \frac{\sqrt{3}}{2}$ *for x in [0, 2π).*

  The triangle given has an angle of 30 degrees, or π/6.

  (Note, if you used your calculator, this is the value you would get)



  Therefore, $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$.

  To answer $\cos x = \frac{\sqrt{3}}{2}$ in the specified domain, we also allow x to be in Quadrant IV,

  so $x = \pi/6$ or $2\pi - \pi/6 = 11\pi/6$.

- *Example. Find the derivative of* $y = \cos^{-1}(e^x + 2x)$.

$$y' = -\frac{1}{\sqrt{1-(e^x+2x)^2}}(e^x+2x)'$$

$$= -\frac{e^x+2}{\sqrt{1-(e^x+2x)^2}}$$

- *Example. Find the derivative of* $y = \sin^{-1}\left(\frac{x+1}{x-1}\right)$

$$y' = \frac{1}{\sqrt{1-\left(\frac{x+1}{x-1}\right)^2}}\left(\frac{x+1}{x-1}\right)'$$
$$= \frac{1}{\sqrt{1-\left(\frac{x+1}{x-1}\right)^2}}\frac{(x-1)-(x+1)}{(x-1)^2}$$
$$= \frac{-2}{(x-1)^2\sqrt{1-\left(\frac{x+1}{x-1}\right)^2}}$$
$$= \frac{-2}{(x-1)\sqrt{(x-1)^2-(x-1)^2\left(\frac{x+1}{x-1}\right)^2}}$$
$$= \frac{-2}{(x-1)\sqrt{(x-1)^2-\left(x+1\right)^2}}$$
$$= \frac{-2}{(x-1)\sqrt{(x^2-2x+1)-\left(x^2+2x+1\right)}}$$
$$= \frac{-2}{(x-1)\sqrt{-4x}}$$
$$= \frac{-1}{(x-1)\sqrt{-x}}$$

Before you can understand the simplification of the derivative, we must look at the domain of the original function, which comes from the fact in order to be defined, $\sin^{-1}\left(\theta\right)$ is $-1 \le \theta \le 1$.

This means that

$\frac{x+1}{x-1} \ge -1$	and	$\frac{x+1}{x-1} \le 1$
$\frac{x+1}{x-1}+1 \ge 0$		$\frac{x+1}{x-1}-1 \le 0$
$\frac{x+1+x-1}{x-1} \ge 0$		$\frac{x+1-(x-1)}{x-1} \le 0$
$\frac{2x}{x-1} \ge 0$		$\frac{2}{x-1} \le 0$
$x > 1$ or $x \le 0$		$x < 1$

So we must take $x \in (-\infty, 0]$.