Section 3.3 – Derivative of Log and Exponential Functions

Recall:

- The relationship between the log and exponential functions: $\log_a x = y \Leftrightarrow a^y = x$.
- Also recall that $\frac{d}{dx}(a^x) = a^x \ln a$.
- If we use implicit differentiation on the equation $a^y = x$ we find

  $$a^y \ln a \cdot y' = 1$$

  $$y' = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}$$

- Therefore, for $y = \log_a x$, $y' = \frac{1}{x \ln a}$.

Derivative of the Exponential Function:

- The derivative of the Natural Exponential Function is itself

  $$\frac{d}{dx}(e^x) = e^x$$

- Q: Why would this be true?

  A: ____.

- Another way (if we don't want to use the $\frac{d}{dx}(a^x) = a^x \ln a$ formula) we need to recall that *e* can be defined by the number so that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$.
- Using the definition of derivative, we find

  $$\frac{d}{dx}(e^x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0}\left[e^x \frac{e^h - 1}{h}\right] = e^x \lim_{h \to 0}\left[\frac{e^h - 1}{h}\right] = e^x(1) = e^x.$$

Derivative of General Exponential:

- Using another way to differentiate $f(x) = a^x$ we need to know three things…
  1. $f(x) = a^x = (e^{\ln a})^x = e^{(\ln a)x}$
  2. $\frac{d}{dx} e^x = e^x$
  3. The chain rule
- We then find that $\frac{d}{dx} a^x = \frac{d}{dx} e^{(\ln a)x} = e^{(\ln a)x}[(\ln a)x]' = (e^{\ln a})^x \ln a = a^x \cdot \ln a$

Derivative of Log Functions:

- From above, the derivative for any log function $y = \log_a x$ is $y' = \dfrac{1}{x \ln a}$.
- For most practical purposes, we use the natural log, or a base of $a = e$. So we have $\dfrac{d}{dx} \ln x = \dfrac{1}{x}$
- So with the chain rule, we say: The derivative of the natural log of a function is one over that function times the derivative of that function.

Examples:

- *Find the derivative of* $y = \ln\left(\dfrac{x}{x+1}\right)$.

- *Find the derivative of* $y = \dfrac{1 + \ln x}{1 + (\ln x)^2}$.

- *Find the derivative and domain for* $f(x) = \dfrac{1}{1+\ln x}$.