Section 3.3 – Derivative of Log and Exponential Functions

Recall:

- The relationship between the log and exponential functions: $\log_a x = y \Leftrightarrow a^y = x$.
- Also recall that $\frac{d}{dx}(a^x) = a^x \ln a$.
- If we use implicit differentiation on the equation $a^y = x$ we find

  $$a^y \ln a \cdot y' = 1$$

  $$y' = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}$$

- Therefore, for $y = \log_a x$, $y' = \frac{1}{x \ln a}$.

Derivative of the Exponential Function:

- The derivative of the Natural Exponential Function is itself

  $$\frac{d}{dx}(e^x) = e^x$$

- Q: Why would this be true?

  A: Well, $\frac{d}{dx}(e^x) = e^x \ln e = e^x$.

- Another way (if we don't want to use the $\frac{d}{dx}(a^x) = a^x \ln a$ formula) we need to recall that *e* can be defined by the number so that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$.
- Using the definition of derivative, we find

  $$\frac{d}{dx}(e^x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \left[e^x \frac{e^h - 1}{h} \right] = e^x \lim_{h \to 0} \left[\frac{e^h - 1}{h} \right] = e^x (1) = e^x.$$

Derivative of General Exponential:

- Using another way to differentiate $f(x) = a^x$ we need to know three things…
  1. $f(x) = a^x = (e^{\ln a})^x = e^{(\ln a)x}$
  2. $\frac{d}{dx} e^x = e^x$
  3. The chain rule
- We then find that $\frac{d}{dx} a^x = \frac{d}{dx} e^{(\ln a)x} = e^{(\ln a)x} [(\ln a)x]' = (e^{\ln a})^x \ln a = a^x \cdot \ln a$

Derivative of Log Functions:

- From above, the derivative for any log function $y = \log_a x$ is $y' = \dfrac{1}{x \ln a}$.
- For most practical purposes, we use the natural log, or a base of $a = e$. So we have $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$
- So with the chain rule, we say: The derivative of the natural log of a function is one over that function times the derivative of that function.

Examples:

- *Find the derivative of* $y = \ln\left(\dfrac{x}{x+1}\right)$.

$$y' = \frac{1}{\dfrac{x}{x+1}}\left(\frac{x}{x+1}\right)'$$
$$= \frac{x+1}{x}\left(\frac{(x+1)(1) - x(1)}{(x+1)^2}\right)$$
$$= \frac{1}{x}\left(\frac{1}{(x+1)}\right)$$
$$= \frac{1}{x(x+1)}$$

- *Find the derivative of* $y = \dfrac{1+\ln x}{1+(\ln x)^2}$.

$$y' = \frac{(1+(\ln x)^2)\left(0+\dfrac{1}{x}\right) - (1+\ln x)\left(0+2(\ln x)\dfrac{1}{x}\right)}{[1+(\ln x)^2]^2}$$
$$= \frac{\dfrac{1+(\ln x)^2}{x} - \dfrac{2(1+\ln x)(\ln x)}{x}}{[1+(\ln x)^2]^2}$$
$$= \frac{1+(\ln x)^2 - 2\ln x - 2(\ln x)^2}{x[1+(\ln x)^2]^2}$$
$$= \frac{1-2\ln x-(\ln x)^2}{x[1+(\ln x)^2]^2}$$

- *Find the derivative and domain for* $f(x) = \dfrac{1}{1+\ln x}$.

  The domain will be restricted in that $x > 0$ (because of ln$x$)
  We require that $1+\ln x \neq 0, \quad \ln x \neq -1, \quad x \neq e^{-1}$

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\begin{aligned}
f'(x) &= [(1+\ln x)^{-1}]' \\
&= -1(1+\ln x)^{-2}\left(\frac{1}{x}\right) \\
&= \frac{-1}{x(1+\ln x)^2}
\end{aligned}
$$