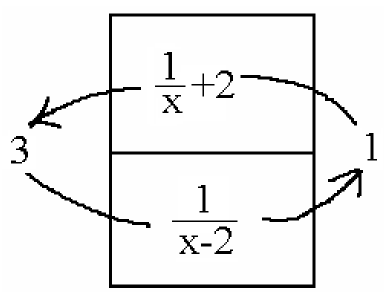
Section 3.2 – Inverse Functions and Logs

Inverse Relations:

- To find an **inverse relation** you are looking to 'undo' the process that was done with the original relation.
- For example, the inverse of the function $f(x) = \frac{1}{x-2}$ is the function $g(x) = \frac{1}{x} + 2$.



Notice that $f(3) = 1$, and $g(1) = 3$.

- Also notice in the previous example, we have the ordered pair (3,1) for $f(x)$ and the ordered pair (1,3) for $g(x)$. In fact, being able to interchange the first and second coordinates of each ordered pair in a relation is another way to recognize an **inverse relation**.
- To find an inverse relation for y = *relation of* x, interchange the x's and y's in the equation and (if possible) solve for y.
- *Example. Find the equation of the inverse of* $y = x^2 - 6x + 9$.

  $x = y^2 - 6y + 9$

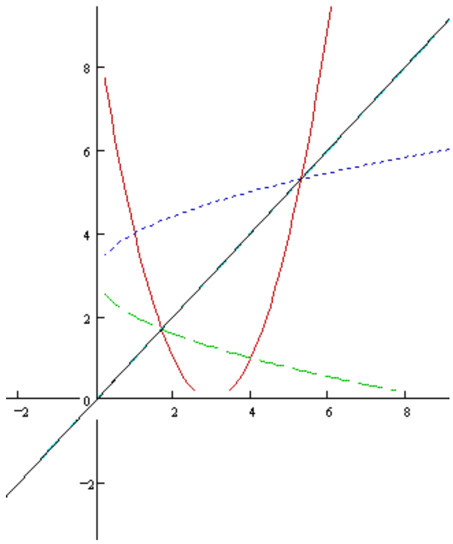
  $x = (y-3)^2$

  $y = \pm\sqrt{x} + 3$
- Q: Is the original problem a function? Does it pass the vertical line test? Is the inverse a function?
  A: Yes, the original problem is a polynomial of degree 2, which is a function. It passes the vertical line test. The inverse, though, is not a function; for each y there are 2 different x's.

Inverse Functions and One-to-One:

- The concepts above hold for functions as well. You can find an inverse function by switching the order of all ordered pairs, or you can switch the x's and y's in the equation and solve.
- The notation for inverse functions is a little bit different, because for a function $f(x)$, whose inverse is also a function, the notation for the inverse is $f^{-1}(x)$.
- Note, this does NOT mean $\frac{1}{f(x)}$, but f inverse!

- Notice in the problem given previously, $f(x) = x^2 - 6x + 9$ is a function. But the inverse, $\pm\sqrt{x} + 3$



is NOT a function.

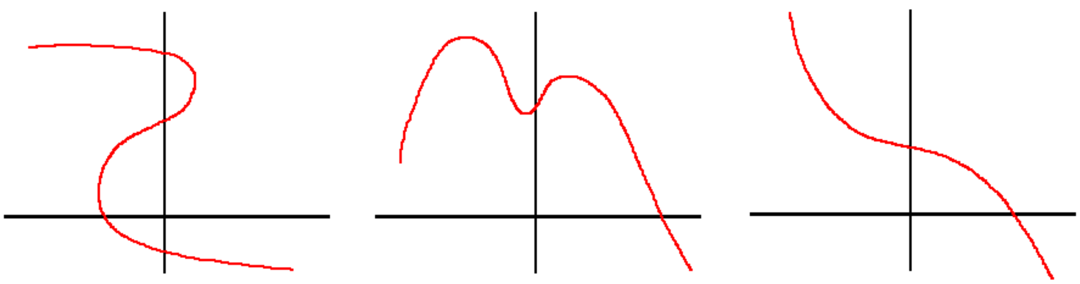
- Q: How can you tell the inverse is not a function?
  A: Looking at the graph, it does not pass the vertical line test. We can't use the f^{-1} notation.

- Recall that for *y*(*x*) to be a function, each *x* has only one *y*.

- *y*(*x*) is **one-to-one** if it is
  1. a function, and
  2. each *y* maps back to only one *x*

- For one-to-one, each *x* has only one *y*, and each *y* has only one *x*. It has to pass not only the vertical line test, but also a **horizontal line test**.

- Q: Which of the following are functions? Which are one-to-one?



a b c

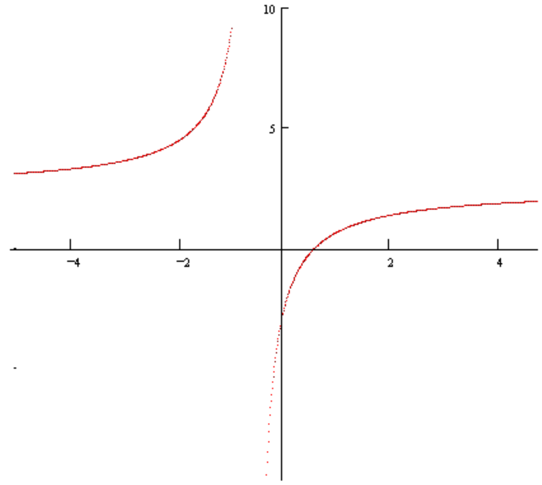
A: a is not a function, so it cannot be 1-1. b is a function (passes the vertical line test) but is not 1-1 (since it fails the horizontal line test). c passes the vertical and horizontal line test, so it's 1-1

- The following functions are always one-to-one:
  Linear, square roots

  The following functions are not ever one-to-one:
  Quadratic, absolute value

- *Example. Graph* $f(x)=\dfrac{5x-3}{2x+1}$*, determine if it is 1-1, if so find the inverse*

  What is the root of $f(x)$?
  When $5x - 3 = 0$, or $x = 3/5$
  What is the undefined value for x?
  When $2x + 1 = 0$, or $x = -1/2$

  It passes the vertical and horizontal line tests.
  It is one to one.

  Find the inverse:

  $$x=\frac{5y-3}{2y+1}$$
  $$2xy+x=5y-3$$
  $$y(2x-5)=-x-3$$
  $$y=\frac{-x-3}{2x-5}=f^{-1}$$

  NOTE: The domain for $f(x)$ is all values except $-1/2$. The range is all values except 5/2
  The domain for $f^{-1}(x)$ is all values except 5/2, and the range is all values except $-1/2$

- Q: What is the relationship between the domain and ranges for functions and their inverses?
  A: They switch, the domain for one is the range for the other, and vice-versa.

- Since an inverse function 'undoes' the original function, if we compose the two [i.e. take $f\circ f^{-1}(x)$ or $f^{-1}\circ f(x)$] we will get x.

- *Example. Show* $f(x)=\dfrac{x+5}{4},\ f^{-1}(x)=4x-5$ *are inverses*

  $$f[f^{-1}(x)]=\frac{[4x-5]+5}{4}$$
  $$=\frac{4x}{4}$$
  $$=x$$

<u>*Log Functions:*</u>

- Recall: $b^y = x$ if and only if $y=\log_b x$
  y is the exponent, b is the base, and x is the argument

- **Special Log Bases:**
  Log base e is **natural log** (written ln)
  Log base 10 is **common log** (written log)

- **Log Rules:**

  $\log_b b = 1$

  $\log_b 1 = 0$

  $\log_b b^p = p$

  $\log_b (MN) = \log_b M + \log_b N$

  $\log_b M^p = p \log_b M$

  $\log_b (M / N) = \log_b M - \log_b N$

  $b^{\log_b p} = p$

- *Example. Express* $\log_a \sqrt{\frac{a^6 b^8}{a^2 b^5}}$ *in terms of sums and differences*

$$\log_a \sqrt{\frac{a^6 b^8}{a^2 b^5}} = \log_a (a^4 b^3)^{1/2}$$
$$= \frac{1}{2} \log_a (a^4 b^3)$$
$$= \frac{1}{2} [\log_a a^4 + \log_a b^3]$$
$$= \frac{1}{2} [4 \log_a a + 3 \log_a b]$$

- *Example, simplify* $2\log_5 x - \log_5 y - 3\log_5 z$

  Since they are all the same base, we can use the rules 'backwards' to combine terms.

$$2\log_5 x - \log_5 y - 3\log_5 z = \log_5 x^2 + \log_5 y^{-1} + \log_5 z^{-3}$$
$$= \log_5 \frac{x^2}{yz^3}$$

<u>*Solving Exponential Equations*</u>:

- Equations with variables in the exponent are called **exponential equations**.
- If the base is the same on both sides of the equation, you can equate the exponents.
- *Example. Solve* $3^{x^2+4x} = \frac{1}{27}$

$$3^{x^2+4x} = 3^{-3}$$
$$x^2 + 4x + 3 = 0$$
$$\therefore \; x = -3, -1$$

- More work comes when the base is not the same. You will then have to solve the equation using logs.

- *Example. Solve* $2^x = 40$

  $\ln(2^x) = \ln(40)$

  $x = \dfrac{\ln 40}{\ln 2}$

- *Example. Solve* $e^x - 6e^{-x} = 1$

  $e^{2x} - e^x - 6 = 0$

  $(e^x - 3)(e^x + 2) = 0$

  So either $e^x = 3 \Rightarrow x = \ln 3$ or $e^x = -2$ (never)

  So $x = \ln 3$

<u>Solving Log Equations</u>:

- Often it is useful to change to exponential in form.
- *Example. Solve* $\log_2 x = -3$

  $2^{-3} = x$

  $x = \dfrac{1}{8}$

- *Example. Solve* $\log_5(8 - 7x) = 3$

  $5^3 = 8 - 7x$

  $7x = -117$

  $x = -\dfrac{117}{7}$

- *Example. Solve* $\log x - \log(x+3) = -1$

  $\log \dfrac{x}{x+3} = -1$

  $10^{-1} = \dfrac{x}{x+3}$

  $0.1(x+3) = x$

  $0.9x = 0.3$

  $x = \dfrac{1}{3}$