Sections 3.1 and 3.4 – Exponential Functions (Growth and Decay)

What Would You Rather Have...

- \$1million, or double your money every day for 31 days starting with 1cent?

Day	Cents	Day	Cents
0	1	16	65536
1	2	17	131072
2	4	18	262144
3	8	19	524288
4	16	20	1048576
5	32	21	2097152
6	64	22	4194304
7	128	23	8388608
8	256	24	16777216
9	512	25	33554432
10	1024	26	67108864
11	2048	27	134217728
12	4096	28	268435456
13	8192	29	536870912
14	16384	30	1073741824
15	32768	31	2147483648

- On day 31, you will have 2,147,483,648 cents, or \$21,474,836.48
- Q: How can we express this with an equation?
  A: $\#cents(day) = 2^{day} \quad \Rightarrow \quad 2^{31} = 2{,}147{,}483{,}648$.
- Q: From the table, on which day will you surpass \$1million, or 10^8 cents?
  A: From the chart of values above, this happens between day 26 and day 27.

Exponential Equations Arise When:

- You add the same percentage to a quantity each fixed time period.
- You multiply a quantity by the same amount each fixed time period.
- For the money example, we were adding 100% each day, or multiplying by 2.

Recall:

- One form of the Exponential Function is $y = k\,b^t$
  b is the base, and *k* is the initial quantity (when $t = 0$).
- Q: What are the conditions on *b*?
  A: *b* has to be positive (greater than zero) and not equal to 1.
- The book uses another form of the exponential equation, $y = a^t$.
  Q: What is different in these forms?
  A: There is no value of *k* in $y = a^t$. Otherwise they are the same format.

- Sometimes we talk about the exponential form of the equation as $y(t) = P_0 e^{kt}$. Be careful of the difference between the forms $P_0\, e^{kt}$ and $k\, b^t$. Sometimes it is standard when one says 'percentage change' that they are explicitly giving the value of *k* in $P_0 e^{kt}$ (as is the case in your textbook) but this typically refers to the change in a population, which is *continuously* growing. It wouldn't be the case for, say, interest which can grow monthly or weekly.
- Q: In the equations $P_0\, e^{kt}$ and $k\, b^t$, what is the relationship between P_0 and *k*, and *k* and *b*?
  A: $P_0 = k$, and $e^k = b$.
- *Rules:*
  $$b^x b^y = b^{x+y}$$
  $$(b^x)^y = b^{xy}$$
  $$b^{-x} = \frac{1}{b^x}$$

The Relationship between Equations:

- When you compare the equations $A\left(1+\frac{r}{n}\right)^{nt}$ and $P_0\, e^{kt}$ and $k\, b^t$, you can see a similarity but be careful which is used.
- Interest problems are of the form $A\left(1+\frac{r}{n}\right)^{nt}$, where *r* is the interest rate, *A* the original amount of money, and *n* is the number of times you compound per year. It is assumed in these problems that *n* is compounded a fixed number of times per year, *not* continuously.
- When money (or a population) grows *continuously* the equation $A\left(1+\frac{r}{n}\right)^{nt}$ changes to $Ae^{r\cdot t}$. To see the relationship here, note that $\lim_{\frac{n}{r}\to\infty}\left(1+\frac{1}{(n/r)}\right)^{n/r} = e$. Your growth rate is then equal to *r*.
- Provided you are dealing with continual growth, you can use either $P_0\, e^{kt}$ or $k\, b^t$. If the growth rate *r* is given,

  Using $k\, b^t$, $b = e^r$

  Using $P_0\, e^{kt}$, $k = r$.

Graphs of Exponentials:

- $y = k\,b^t$

  − If the base is greater than 1, you have **exponential growth**.
  − If the base is less than 1, you have **exponential decay**.

  − The y intercept is k.

  − If $k > 0$, the function will always be positive. As t gets large, the function will tend to 0 (for decay) and infinity (for growth).

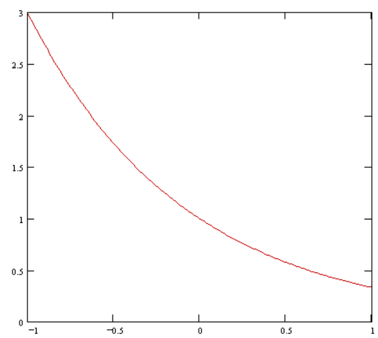
  − If $k < 0$, the function will always be negative (it is flipped upside down).

- If you have an equation that is exponential in form, but are adding or subtracting a constant
  $y = k\,b^t + c$, this is a shift up ($c > 0$) or down ($c < 0$) by c
  $y = k\,b^{(t+c)}$, this is a shift left ($c < 0$) or right ($c > 0$) by c

- *Example. Graph* $y = 3^{-x}$

  $$y = \left(\frac{1}{3}\right)^x$$

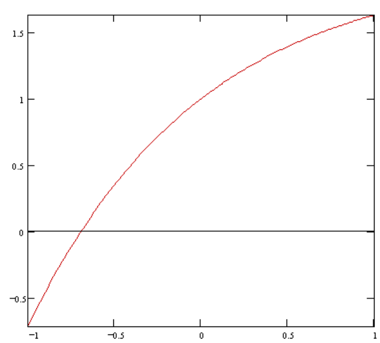
  $b = 1/3$
  Exponential decay
  $k = 1$
  Crosses at (0,1)



- *Example. Graph* $f(x) = 2 - e^{-x}$

  $$f(x) = 2 + (-1)\left(\frac{1}{e}\right)^x$$

  This has a shift up of 2
  $k = -1$ (flipped upside down)
  and $b = \frac{1}{e} \approx 0.37$



Testing a Data Set (OPTIONAL):

t	y	y2 / y1
0	2	
1	6	3
2	18	3
3	54	3
4	162	3

- Determine if the following set of data is exponential by taking divisions of successive y values
- For each fixed time period (change in t is one) we seem to be multiplying by 3.
- This means that $b = (y2 / y1) / (\text{change in } t) = 3$, and $k = y(0) = 2$.
- $y(t) = 2 \cdot 3^t$

Another Example, Simple Interest (OPTIONAL):

- Suppose your savings account earns 1.25% interest per month, and you start with \$250. Write an equation that represents this problem.

Month	Formula	Dollars
0	250 =	250.00
1	250 + 1.25%(250) =	253.13
2	253.13 + 1.25%(253.13) =	256.29
3	256.29 + 1.25%(256.29) =	259.49
4	259.49 + 1.25%(259.49) =	262.73
5	262.74 + 1.25%(262.74) =	266.01

- How can we make an equation out of this?

  $$y(t) = 250(1.0125)^t$$

- Q: Does this represent growth or decay?
  A: Growth, because the dollar value is getting larger.

- Q: How much do you *earn* in the first month?
  A: $y(1) - 250 = 250(1.0125) - 250 \approx 3.13$

Half Life and Doubling Time:

- **Half life** is the amount of time it takes for a substance to decay to half of its original quantity. It represents exponential decay.

- The half life (T) and decay rate (b) are related

  $$\frac{1}{2}k = k \cdot b^T$$

  $$T = \frac{\ln(0.5)}{\ln b} \quad \text{or} \quad b = \left(\frac{1}{2}\right)^{1/T}$$

- **Doubling time** is the amount of time it takes for a substance to double its original quantity. It represents exponential growth.

- The doubling time (T) and growth rate (b) are also related

  $$2k = k \cdot b^T$$

  $$T = \frac{\ln(2)}{\ln b} \quad \text{or} \quad b = 2^{1/T}$$

- You can either memorize the equations above, or use your knowledge of the problem and equation to determine the values needed.

- *Example. If a species of bacteria has a doubling time of 45 minutes, then find b in* $k\,b^t$.

  $y = kb^t$

  $2k = kb^{45}$

  $2 = b^{45}$

  $b = 2^{1/45} \approx 1.015523$

- *Example. If a species of bacteria has a doubling time of 45 minutes, then find k in* $P_0\,e^{k\,t}$.

  $y = P_0 e^{kt}$

  $2P_0 = P_0 e^{k(45)}$

  $2 = e^{k(45)}$

  $k = \dfrac{\ln(2)}{45} \approx 0.015$

- *Example: The world population was 2560 million in 1950 and 3040 million in 1960. Model the population with an exponential equation.*

  Let $t = 0$ represent 1950 (this makes the 'math' easier).

  Method 1, using $k\,b^t$

  $2560 = kb^0 \Rightarrow k = 2560$

  $P = 2560b^t$

  $3040 = 2560b^{10} \Rightarrow b = (3040/2560)^{1/10} \approx 1.0173$

  $P = 2560(1.0173)^t$

  Method 2, using $P_0\,e^{k\,t}$

  $2560 = P_0 e^{k\cdot 0} \Rightarrow P_0 = 2560$

  $P = 2560e^{kt}$

  $3040 = 2560e^{k\cdot 10} \Rightarrow k = \dfrac{\ln(3040/2560)}{10} \approx 0.0172$

  $P = 2560e^{(0.0172)t}$

  Note that $e^{(0.0172)} = 1.0173$