

## Section 2.5 – The Chain Rule

### Our Limitations:

- So far we have only looked at simple functions, products and quotients. i.e.  $f(x) = \frac{2x+1}{x^2+3}$ .
- **Q: For our skill set as it stands, how would we find the derivative of  $f(x) = (2x+1)^2$ ?**  
**A: We would have to use the product rule  $(2x+1)(2x+1)$  or multiply it out.**
- This doesn't seem so bad, but what if we wanted to differentiate  $f(x) = (2x+1)^{12}$ . We would prefer another, quicker, way.
- In a similar way, we don't have the tools to differentiate  $f(x) = \sqrt{4x+1}$ , or  $f(x) = \sin(x^2 + e^x)$
- What is so special about these functions above? They are **composite functions**, meaning you have one function inside another function.
- That is,  $f(x) = \sqrt{4x+1}$  can be thought of as the function  $4x+1$  *inside* the function  $\sqrt{x}$ .

### The Chain Rule:

- The chain rule is what you use when differentiating composite functions. This is why we memorized all of our previous rules as text instead of symbols. Let's review them, the derivative of...
  - ...a constant is zero
  - ...a function to a power is the power times the function to the one less power
  - ...a sum is the sum of the derivatives
  - ...a constant times a function is the constant times the derivative of that function
  - ...the Natural Exponential Function is itself
  - ...a product is the 1st times the derivative of the 2nd plus the 2nd times the derivative of the 1st
  - ...a quotient is the bottom times the derivative of the top minus the top times the derivative of the bottom, all divided by the bottom squared
  - ...sine is cosine
  - ...cosine is minus sine
  - ...tangent is secant squared
  - ...cosecant is minus cosecant times cotangent
  - ...secant is secant times tangent
  - ...cotangent is minus cosecant squared
- All the chain rule says is that you append to the end "times the derivative of that function"
- For example, the derivative of a function to a power is the power times the function to the one less power times the derivative of that function.
- *Example. Find the derivative of  $f(x) = (2x+1)^{12}$*

First, you must realize that you have (overall) a function  $(2x+1)$  to a power (12)

$$f'(x) = 12(2x+1)^{12-1}(2x+1)' = 12(2x+1)^{11}(2) = 24(2x+1)^{11}.$$

- *Example. Find the derivative of  $f(x) = \sqrt{4x+1}$*

$$\begin{aligned} f'(x) &= \frac{d}{dx}(4x+1)^{1/2} \\ &= \frac{1}{2}(4x+1)^{\frac{1}{2}-1} (4x+1)' \\ &= \frac{1}{2}(4x+1)^{-\frac{1}{2}} (4) \\ &= \frac{2}{\sqrt{4x+1}} \end{aligned}$$

- As another example, the derivative of the sine of a function is the cosine of that function times the derivative of that function.
- *Example. Find the derivative of  $f(x) = \sin(x^2 + e^x)$*

Again, realize that overall you have the sine of a function

$$\begin{aligned} f'(x) &= \cos(x^2 + e^x) (x^2 + e^x)' \\ &= \cos(x^2 + e^x) (2x + e^x) \\ &= (2x + e^x) \cos(x^2 + e^x) \end{aligned}$$

*Practice Makes Perfect:*

- Realize that you have to practice this to get it down. It may not seem natural to you at first.
- The other thing that is essential for success is that you must be able to understand what type of function you have *overall*.
- Let's work on this...

$f(x) = \frac{x}{\sqrt{x^3 + 2x}}$  is first a quotient. Then you have a function to a power on the bottom.

$g(x) = \cos(2x) \cdot \sin(x^2 + 2x)$  is first a product, then you have both sine and cosine functions with functions inside each.

$h(x) = (x^2 + 1) + \tan(e^x)$  is first a sum. Then you have tangent with a function inside.

$k(x) = \sqrt{x + \sqrt{x + \sqrt{1}}}$  is first a function to a power. Then you have another function to a power. Don't let the  $\sqrt{1}$  confuse you, it is just a constant.

- Let's differentiate these...

$$f'(x) = \frac{\sqrt{x^3 + 2x}(1) - x \frac{1}{2}(x^3 + 2x)^{-1/2}(3x)}{(x^3 + 2x)}$$

$$g(x) = \cos(2x) \cdot \cos(x^2 + 2x) \cdot (2x + 2) + \sin(x^2 + 2x) \cdot (-\sin(2x)) \cdot 2$$

$$h'(x) = 2x + 0 + \sec^2(e^x) \cdot e^x$$

$$k(x) = (x + [x + 1]^{1/2})^{1/2}$$

$$k'(x) = \frac{1}{2}(x + [x + 1]^{1/2})^{-1/2} [x + [x + 1]^{1/2}]' = \frac{1}{2}(x + [x + 1]^{1/2})^{-1/2} [1 + \frac{1}{2}[x + 1]^{-1/2}(1)]$$

- Note I did not 'clean up' any of these problems so you could clearly see how the chain rule works.

Derivative of General Exponential:

- In order to differentiate  $f(x) = a^x$  we need to know three things...

1.  $a^x = (e^{\ln a})^x = e^{(\ln a)x}$

2.  $\frac{d}{dx} e^x = e^x$

3. The chain rule

- The derivative of a general exponential is

$$\begin{aligned} \frac{d}{dx} a^x &= \frac{d}{dx} e^{(\ln a)x} \\ &= e^{(\ln a)x} [(\ln a)x]' \\ &= (e^{\ln a})^x [\ln a] \\ &= a^x \cdot \ln a \end{aligned}$$