Section 2.4 – The Product and Quotient Rules

Product Rule:

- Let's look at the limit function of the derivative of a product…

$$\frac{d}{dx}[f(x)\cdot g(x)] = \lim_{h\to 0}\frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h}$$
$$= \lim_{h\to 0}\frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x) \quad + \quad [f(x+h)\cdot g(x) - f(x+h)\cdot g(x)]}{h}$$
$$= \lim_{h\to 0}\frac{f(x+h)\cdot g(x+h) - f(x+h)\cdot g(x) - f(x)\cdot g(x) + f(x+h)\cdot g(x)}{h}$$
$$= \lim_{h\to 0}\frac{f(x+h)[\ g(x+h) - g(x)\ ] \quad + \quad g(x)[\ f(x+h) - f(x)\ ]}{h}$$
$$= f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

- So in general, $\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot g'(x) + g(x)\cdot f'(x)$
- The derivative of a product $f(x)\cdot g(x)$ is… the first times the derivative of the second plus the second times the derivative of the first
- Remember, it is essential for future success that you memorize the words and not the symbolic formula.

Quotient Rule:

- In general, you can think of a quotient as a product…

$$\frac{d}{dx}\left[f(x)\cdot\frac{1}{g(x)}\right] = f(x)\cdot\frac{d}{dx}\left(\frac{1}{g(x)}\right) + \frac{1}{g(x)}\cdot f'(x)$$

Now, we don't actually have the tools yet to find $\frac{d}{dx}\left(\frac{1}{g(x)}\right) = \frac{-g'(x)}{[g(x)]^2}$,

but let's take it on faith (for now) and plug it in and simplify

$$\frac{d}{dx}\left[f(x)\cdot\frac{1}{g(x)}\right] = f(x)\cdot\frac{-g'(x)}{[g(x)]^2} + \frac{1}{g(x)}\cdot f'(x)$$
$$= \frac{-f(x)\cdot g'(x)}{[g(x)]^2} + \frac{g(x)\cdot f'(x)}{[g(x)]^2}$$
$$= \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2}$$

- So in general we have, $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{g(x)\cdot f'(x)-f(x)\cdot g'(x)}{[g(x)]^2}$
- The derivative of a quotient $\dfrac{f(x)}{g(x)}$ is…
  the bottom times the derivative of the top minus the top times the derivative of the bottom, all divided by the bottom squared

Some Example Problems:

- *Example. Differentiate* $f(x)=(x+1)(x-2)$
- *Example. Differentiate* $g(x)=\dfrac{x}{x^2+1}$
- *Example. Differentiate* $h(x)=\dfrac{x^2}{1+\dfrac{1}{x}}$

- *Example. Find the equation of the tangent line to the curve* $y(x) = x^{2/3}(x+1)$ *at* $x = 1$.

Derivatives of Other Trig Functions:

- We can find the derivative of all other trig functions using our knowledge of how they relate back to sine and cosine.
- Derivative of tangent

$$\begin{aligned} \frac{d}{dx}\tan(x) &= \frac{d}{dx}\frac{\sin(x)}{\cos(x)} \\ &= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \sec^2 x \end{aligned}$$

The derivative of tangent is secant squared

- Derivative of cosecant

$$\begin{aligned} \frac{d}{dx}(\csc x) &= \frac{d}{dx}\left(\frac{1}{\sin x}\right) \\ &= \frac{\sin x(0) - 1(\cos x)}{\sin^2 x} \\ &= \frac{1}{\sin x}\frac{-\cos x}{\sin x} \\ &= -\csc x \cot x \end{aligned}$$

The derivative of cosecant is minus cosecant times cotangent

- Derivative of secant

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$$
$$= \frac{\cos x(0) - 1(-\sin x)}{\cos^2 x}$$
$$= \frac{1}{\cos x}\frac{\sin x}{\cos x}$$
$$= \sec x \tan x$$

The derivative of secant is secant times tangent

- Derivative of cotangent

$$\frac{d}{dx}\cot x = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$$
$$= \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$
$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$
$$= -\csc^2 x$$

The derivative of cotangent is minus cosecant squared

- *Example. Find the derivative of* $f(x) = e^x \sec x$

- *Example. Find the derivative of* $f(x) = \dfrac{1}{x + \tan x}$

NOTE: this may not be in *simple* terms, but has demonstrated differentiation completely.