

## Section 2.4 – The Product and Quotient Rules

### Product Rule:

- Let's look at the limit function of the derivative of a product...

$$\begin{aligned} \frac{d}{dx}[f(x) \cdot g(x)] &= \lim_{h \rightarrow 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x) + [f(x+h) \cdot g(x) - f(x+h) \cdot g(x)]}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) \cdot g(x+h) - f(x+h) \cdot g(x) - f(x) \cdot g(x) + f(x+h) \cdot g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]}{h} \\ &= f(x) \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} + g(x) \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f(x) \cdot g'(x) + g(x) \cdot f'(x) \end{aligned}$$

- So in general,  $\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot g'(x) + g(x) \cdot f'(x)$
- The derivative of a product  $f(x) \cdot g(x)$  is... **the first times the derivative of the second plus the second times the derivative of the first**
- Remember, it is essential for future success that you memorize the words and not the symbolic formula.

### Quotient Rule:

- In general, you can think of a quotient as a product...

$$\frac{d}{dx} \left[ f(x) \cdot \frac{1}{g(x)} \right] = f(x) \cdot \frac{d}{dx} \left( \frac{1}{g(x)} \right) + \frac{1}{g(x)} \cdot f'(x)$$

Now, we don't actually have the tools yet to find  $\frac{d}{dx} \left( \frac{1}{g(x)} \right) = \frac{-g'(x)}{[g(x)]^2}$ ,

but let's take it on faith (for now) and plug it in and simplify

$$\begin{aligned} \frac{d}{dx} \left[ f(x) \cdot \frac{1}{g(x)} \right] &= f(x) \cdot \frac{-g'(x)}{[g(x)]^2} + \frac{1}{g(x)} \cdot f'(x) \\ &= \frac{-f(x) \cdot g'(x)}{[g(x)]^2} + \frac{g(x) \cdot f'(x)}{[g(x)]^2} \\ &= \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2} \end{aligned}$$

- So in general we have,  $\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$
- The derivative of a quotient  $\frac{f(x)}{g(x)}$  is...

the bottom times the derivative of the top minus the top times the derivative of the bottom, all divided by the bottom squared

Some Example Problems:

- *Example. Differentiate*  $f(x) = (x+1)(x-2)$

We 'could' multiply this out, and use the tools from the last section

$$f'(x) = (x^2 - x - 2)' = 2x - 1$$

Or we could use the product rule...

$$f'(x) = (x+1) \cdot (x-2)' + (x-2) \cdot (x+1)' = (x+1)(1) + (x-2)(1) = 2x - 1$$

- *Example. Differentiate*  $g(x) = \frac{x}{x^2+1}$

$$\begin{aligned} g'(x) &= \frac{(x^2+1)(x)' - (x)(x^2+1)'}{(x^2+1)^2} \\ &= \frac{(x^2+1)(1) - (x)(2x)}{(x^2+1)^2} \\ &= \frac{x^2+1-2x^2}{(x^2+1)^2} \\ &= \frac{1-x^2}{(x^2+1)^2} \end{aligned}$$

- *Example. Differentiate*  $h(x) = \frac{x^2}{1+\frac{1}{x}}$

$$\begin{aligned} h'(x) &= \left( \frac{x^2}{1+x^{-1}} \right)' = \frac{(1+x^{-1})(x^2)' - (x^2)(1+x^{-1})'}{(1+x^{-1})^2} \\ &= \frac{(1+x^{-1})(2x) - (x^2)(-1x^{-2})}{(1+x^{-1})^2} \\ &= \frac{(2x+2)+1}{(1+x^{-1})^2} \\ &= \frac{2x+3}{(1+x^{-1})^2} = \frac{2x+3}{\left(1+\frac{1}{x}\right)^2} \end{aligned}$$

- *Example. Find the equation of the tangent line to the curve  $y(x) = x^{2/3}(x+1)$  at  $x = 1$ .*

$$y(1) = 2$$

$$y'(x) = x^{2/3}(x+1)' + (x+1)(x^{2/3})'$$

$$= x^{2/3}(1) + (x+1)\left(\frac{2}{3}x^{-1/3}\right)$$

$$y'(1) = (1)^{2/3} + (2+1)\left(\frac{2}{3}(1)^{-1/3}\right)$$

$$= 1 + (3)\left(\frac{2}{3}\right)$$

$$= \frac{7}{3}$$

$$y - 2 = \frac{7}{3}(x - 1)$$

Derivatives of Other Trig Functions:

- We can find the derivative of all other trig functions using our knowledge of how they relate back to sine and cosine.
- Derivative of tangent

$$\begin{aligned} \frac{d}{dx} \tan(x) &= \frac{d}{dx} \frac{\sin(x)}{\cos(x)} \\ &= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \sec^2 x \end{aligned}$$

The derivative of tangent is secant squared

- Derivative of cosecant

$$\begin{aligned} \frac{d}{dx} (\csc x) &= \frac{d}{dx} \left( \frac{1}{\sin x} \right) \\ &= \frac{\sin x(0) - 1(\cos x)}{\sin^2 x} \\ &= \frac{1}{\sin x} \frac{-\cos x}{\sin x} \\ &= -\csc x \cot x \end{aligned}$$

The derivative of cosecant is minus cosecant times cotangent

- Derivative of secant

$$\begin{aligned}\frac{d}{dx}(\sec x) &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) \\ &= \frac{\cos x(0) - 1(-\sin x)}{\cos^2 x} \\ &= \frac{1 \sin x}{\cos x \cos x} \\ &= \sec x \tan x\end{aligned}$$

The derivative of secant is secant times tangent

- Derivative of cotangent

$$\begin{aligned}\frac{d}{dx}\cot x &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) \\ &= \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x} \\ &= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\ &= -\csc^2 x\end{aligned}$$

The derivative of cotangent is minus cosecant squared

- *Example.* Find the derivative of  $f(x) = e^x \sec x$

$$\begin{aligned}f'(x) &= e^x(\sec x)' + \sec x(e^x)' \\ &= e^x(\sec x \tan x) + \sec x(e^x) \\ &= e^x \sec x(1 + \tan x)\end{aligned}$$

- *Example.* Find the derivative of  $f(x) = \frac{1}{x + \tan x}$

$$\begin{aligned}f'(x) &= \frac{(x + \tan x)(0) - 1(1 + \sec^2 x)}{(x + \tan x)^2} \\ &= \frac{-(1 + \sec^2 x)}{(x + \tan x)^2}\end{aligned}$$

NOTE: this may not be in *simple* terms, but has demonstrated differentiation completely.