Section 2.1 – Derivatives and Rates of Change

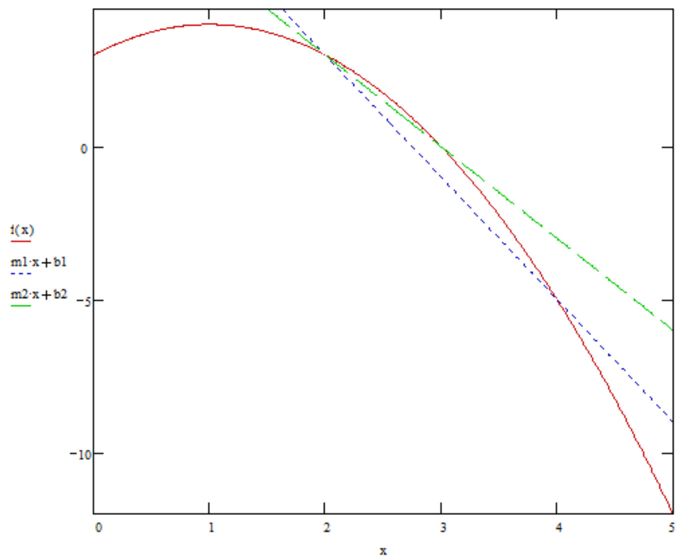
Recap:

- Lets look at the function $f(x) = -x^2 + 2x + 3$.

  Q: What is the equation of the line through the points on the graph at $x = 2$ and $x = 4$?
  A: $f(2) = 3$ and $f(4) = -5$
  So we have (2,3) and (4, –5)
  $m = -4$. $b = f(2) - 2m = 11$
  The equation: $y1 = -4x + 11$

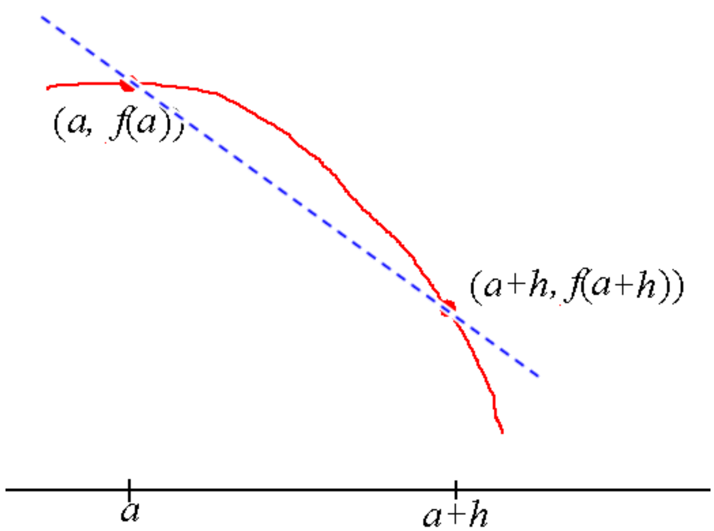
  Q: What about the equation between 2 and 3?
  A: $f(2) = 3$ and $f(3) = 0$
  So we have (2,3) and (3,0)
  $m = -3$. $b = f(2) - 2m = 9$
  The equation: $y2 = -3x + 9$

  Q: Can you predict the tangent line at $x = 2$?
  A: It would have a slope of –2, and go through the point (2,3). $y = -2x + 7$



The General Process:

- From an original starting point (a, $f(a)$) we are looking at the slope of the line connecting it to another point a distance of h away from a.



- What is the slope of this dashed line connecting (a, $f(a)$) and ($a+h$, $f(a+h)$)?

$$\Delta y = f(a+h) - f(a)$$

$$\Delta x = (a+h) - a = h$$

$$slope = \frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}$$

- We want to let h tend to zero, so that the point ($a+h$, $f(a+h)$) collapses into (a, $f(a)$). So we take $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.
- This is slope of the tangent line, also called the instantaneous rate of change of $f(x)$ at $x = a$.

Same Definition, Different Names:

- The formula given by $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ has a few different names.
- The **slope of the tangent line** to the curve $f(x)$ at the point a.
- The **instantaneous velocity** of the function $f(x)$ at the point a.
- The **derivative** of $f(x)$ at the point a.
- The **instantaneous rate of change** of $f(x)$ at the point a.
- If $f(x)$ represents distance at any time, then the above definition is the **velocity** at the point a. If we take the absolute value of this, it represents **speed**.

Some Examples:

- *Find the slope of the tangent line to the parabola* $y = x^2 - 3x$ *at the point* $x = -1$.

$$\begin{aligned}
\lim_{h \to 0} \frac{y(-1+h) - y(-1)}{h} &= \lim_{h \to 0} \frac{[(-1+h)^2 - 3(-1+h)] - [(-1)^2 - 3(-1)]}{h} \\
&= \lim_{h \to 0} \frac{1 - 2h + h^2 + 3 - 3h - 4}{h} \\
&= \lim_{h \to 0} \frac{-5h + h^2}{h} \\
&= \lim_{h \to 0} (-5 + h) \\
&= -5
\end{aligned}$$

So the slope of the tangent line at the given point is –5.

- *Find an equation of the tangent line to the curve* $y=\sqrt{5x-4}$ *at the point x = 4.*

  First we see that $y(4) = \sqrt{5(4)-4}=4$.

$$\lim_{h\to 0}\frac{y(4+h)-y(4)}{h}=\lim_{h\to 0}\frac{\sqrt{5(4+h)-4}-4}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{16+5h}-4}{h}\cdot\frac{\sqrt{16+5h}+4}{\sqrt{16+5h}+4}$$
$$=\lim_{h\to 0}\frac{16+5h-16}{h(\sqrt{16+5h}+4)}$$
$$=\lim_{h\to 0}\frac{5}{\sqrt{16+5h}+4}$$
$$=\frac{5}{8}$$

  The tangent line has a slope of 5/8 and goes through the point (4,4).

  The equation is given by $y-4=\frac{5}{8}(x-4)$.

- *Find the slope of the tangent to the curve* $y=\frac{1}{x-3}$ *at any point a.*

$$\lim_{h\to 0}\frac{y(a+h)-y(a)}{h}=\lim_{h\to 0}\frac{\frac{1}{a+h-3}-\frac{1}{a-3}}{h}$$
$$=\lim_{h\to 0}\frac{\frac{a-3}{(a+h-3)(a-3)}-\frac{a+h-3}{(a+h-3)(a-3)}}{h}$$
$$=\lim_{h\to 0}\frac{a-3-(a+h-3)}{h(a+h-3)(a-3)}$$
$$=\lim_{h\to 0}\frac{-h}{h(a+h-3)(a-3)}$$
$$=\lim_{h\to 0}\frac{-1}{(a+h-3)(a-3)}$$
$$=\frac{1}{(a-3)^2}$$

- *Find the slope of the tangent to the curve* $y = 2\sqrt{x}$. *This time we will leave it as x instead of a.*

$$\lim_{h\to 0}\frac{y(x+h)-y(x)}{h} = \lim_{h\to 0}\frac{2\sqrt{x+h}-2\sqrt{x}}{h}$$

$$= \lim_{h\to 0}\frac{2\sqrt{x+h}-2\sqrt{x}}{h}\cdot\frac{2\sqrt{x+h}+2\sqrt{x}}{2\sqrt{x+h}+2\sqrt{x}}$$

$$= \lim_{h\to 0}\frac{4(x+h)-4x}{h(2\sqrt{x+h}+2\sqrt{x})}$$

$$= \lim_{h\to 0}\frac{4h}{h(2\sqrt{x+h}+2\sqrt{x})}$$

$$= \frac{4}{4\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}}$$