Section 1.6 – Limits at Infinity

Recall:

- You have already studied **rational functions**, that is $f(x)=\frac{p(x)}{q(x)}$.
- You should already be able to identify the domain of this function, in other words, the values of *x* that are/are not allowed for inputs of *f*.
- Q: How do we find any undefined points in the domain?
  A: Set the denominator equal to zero and solve for *x*.
- These points, if they are not **removable discontinuities**, are called **infinite discontinuities**, and the limit of the function there would tend to plus or minus infinity.
- These types of infinite discontinuities are also called **vertical asymptotes**.
- *Example. For* $f(x)=\frac{1}{x-2}$ *What is the limit as x tends to 2 from the right? From the left?*

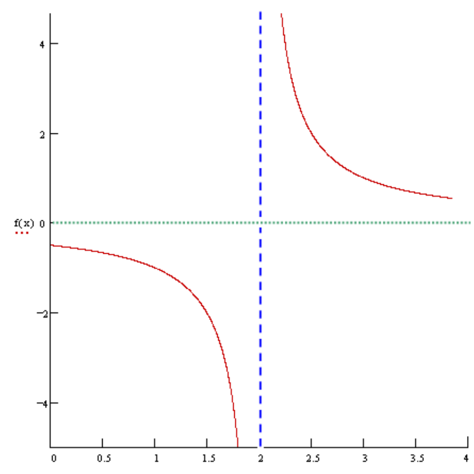
  First note that the domain is restricted when
  $x-2=0 \Rightarrow x=2$.

  Also note that the top does not go to 0 at *x* = 2.

  x = 2 is an infinite discontinuity.

  $\lim_{x\to 2^+} f(x)=\infty$

  $\lim_{x\to 2^-} f(x)=-\infty$



Horizontal Asymptotes:

- Think of an asymptote in general being an invisible line that a function 'tends' to if you were to keep drawing. You never quite get there.
- For example, in the above function, the graph (as *x* tends to 2) gets closer and closer to the vertical line at *x* = 2 but never quite reaches it. And it will not cross over, either.
- The above example function has another asymptote that is horizontal… If you were to let *x* get larger and larger ($x\to\pm\infty$) the function would get closer and closer to an 'invisible' horizontal line.
- Q: What value does the function tend to as $x\to\pm\infty$? What is the equation of this line?
  A: It tends to 0, or the equation *y* = 0.

Finding Horizontal Asymptotes:

- As we 'extend' the function to the left and right as far as we want, we are essentially taking the limit as x tends to plus and minus infinity.
- To find the horizontal asymptote, if it exists, take the limit as x tends to plus and minus infinity.
- **HOW TO** find horizontal asymptotes for rational functions $f(x) = \frac{p(x)}{q(x)}$:

  1. Find the leading term of the numerator, $p(x)$ and denominator, $q(x)$.
  2. Write as a quotient and simplify.
  3. If the reduction is
     – a constant, then this is the value of your horizontal asymptote.
     – $\text{constant} \cdot \left(\frac{1}{x^{\text{positive value}}} \right)$, then the horizontal asymptote is zero
     – $\text{constant} \cdot (x^{\text{positive value}})$, then there is no horizontal asymptote
     It will tend to plus or minus infinity (plug in to see which).

- *Example. Find* $\lim_{x\to\infty} \frac{3x^2+x+5}{x^2-4}$

  The leading term of the top is $3x$. The leading term of the bottom is x.

  So we have $\frac{3x^2}{x^2} = 3$.

  This problem has a horizontal asymptote at $y = 3$.

  $\lim_{x\to\infty} \frac{3x^2+x+5}{x^2-4} = 3$.

  Note that $\lim_{x\to-\infty} \frac{3x^2+x+5}{x^2-4} = 3$.

- *Example. Find* $\lim_{x\to\infty} \frac{3x^2+x^3+5}{2x^2+x^4-4}$

  The leading term of the top is x^3. The leading term of the bottom is x^4.

  So we have $\frac{x^3}{x^4} = \frac{1}{x}$.

  This problem has a horizontal asymptote at $y = 0$.

  $\lim_{x\to\infty} \frac{3x^2+x^3+5}{2x^2+x^4-4} = 0$.

  Note that $\lim_{x\to-\infty} \frac{3x^2+x^3+5}{2x^2+x^4-4} = 0$.

- *Example. Find* $\lim_{x\to\infty}\dfrac{3x^2+x^3+5}{2x^2-4}$

  The leading term of the top is x^3. The leading term of the bottom is $2x^2$

  So we have $\dfrac{x^3}{2x^2}=\dfrac{1}{2}x$.

  This problem has no horizontal asymptote.

  $$\lim_{x\to\infty}\frac{3x^2+x^3+5}{2x^2-4}=\lim_{x\to\infty}\frac{1}{2}x$$
  $$=+\infty$$

- *Example. Find* $\lim_{x\to-\infty}\dfrac{x^2(x-1)(x+2)}{3-x}$

  The leading term of the top is $x^2(x)(x)=x^4$. The leading term of the bottom is $-x$.

  So we have $\dfrac{x^4}{-x}=-x^3$.

  There is no horizontal asymptote.

  $$\lim_{x\to-\infty}\frac{x^2(x-1)(x+2)}{3-x}=\lim_{x\to-\infty}(-x^3).$$
  $$=+\infty$$