Section 1.4 – Calculating Limits

- In this section we will be using methods to find limits, not prove them as in the last section.
- Sometimes it is helpful to 'break up' limits and work with pieces rather than the whole.
- *Example. Find* $\lim_{x\to 2}\left(2+\frac{1}{(x-2)^2}\right)$

  $\lim_{x\to 2} 2 = 2$ and $\lim_{x\to 2}\frac{1}{(x-2)^2} = +\infty$

  Therefore, $\lim_{x\to 2}\left(2+\frac{1}{(x-2)^2}\right)=\infty$

- Most of these rules are intuitive, in that you would think of using them anyway because you are familiar with the rules of addition, subtraction, multiplication and division.

  $\lim_{x\to a} c = c$

  $\lim_{x\to a} x^n = a^n$

  $\lim_{x\to a}\sqrt[n]{x}=\sqrt[n]{a}$ $\quad a>0$ for n even

  $\lim_{x\to a}\left[f(x)\pm g(x)\right]=\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x)$

  $\lim_{x\to a} c\cdot f(x)=c\cdot\lim_{x\to a} f(x)$

  $\lim_{x\to a}\left[f(x)\cdot g(x)\right]=\lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$

  $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ $\quad \lim_{x\to a} g(x)\neq 0$

  $\lim_{x\to a}\left[f(x)\right]^n=\left[\lim_{x\to a} f(x)\right]^n$

  $\lim_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim_{x\to a} f(x)}$ $\quad \lim_{x\to a} f(x)>0$ for n even

- A very useful theorem is called the **squeeze theorem**, and it says that if $f(x)\le g(x)\le h(x)$ when x is close to a and $\lim_{x\to a} f(x)=\lim_{x\to a} h(x)=L$, then $\lim_{x\to a} g(x)=L$.

  Essentially what is happening is that *f* and *h* are tending to *L*, and since *g* is squeezed in between them, it must also tend to *L*.

Example Problems:

- *Example. Evaluate* $\lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}$

$$\lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}=\lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}\left(\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}\right)$$
$$=\lim_{h\to 0}\frac{(1+h)-1}{h(\sqrt{1+h}+1)}$$
$$=\lim_{h\to 0}\frac{h}{h(\sqrt{1+h}+1)}$$
$$=\lim_{h\to 0}\frac{1}{\sqrt{1+h}+1}$$
$$=\frac{1}{2}$$

- *Example. Evaluate* $\lim_{h\to 0}\frac{(3+h)^{-1}-3^{-1}}{h}$

$$\lim_{h\to 0}\frac{(3+h)^{-1}-3^{-1}}{h}=\lim_{h\to 0}\frac{\frac{1}{3+h}-\frac{1}{3}}{h}\cdot\frac{3\cdot(3+h)}{3\cdot(3+h)}$$
$$=\lim_{h\to 0}\frac{3-(3+h)}{3h\cdot(3+h)}$$
$$=\lim_{h\to 0}\frac{-1}{3\cdot(3+h)}$$
$$=\frac{-1}{9}$$

- *Example. Evaluate* $\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)$

$$\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)=\lim_{t\to 0}\frac{(t^2+t)-t}{t(t^2+t)}$$
$$=\lim_{t\to 0}\frac{t}{(t^2+t)}$$
$$=\lim_{t\to 0}\frac{1}{t+1}$$
$$=1$$

- *Example. Evaluate* $\lim\limits_{x\to 2}\dfrac{x^2-4}{|x-2|}$

  as $x \to 2^+$, $|x-2| = x-2$.

  So $\lim\limits_{x\to 2^+}\dfrac{x^2-4}{|x-2|} = \lim\limits_{x\to 2^+}\dfrac{(x+2)(x-2)}{x-2}$

  $= \lim\limits_{x\to 2^+}\dfrac{x+2}{1} = 4$

  But as $x \to 2^-$, $|x-2| = -(x-2)$.

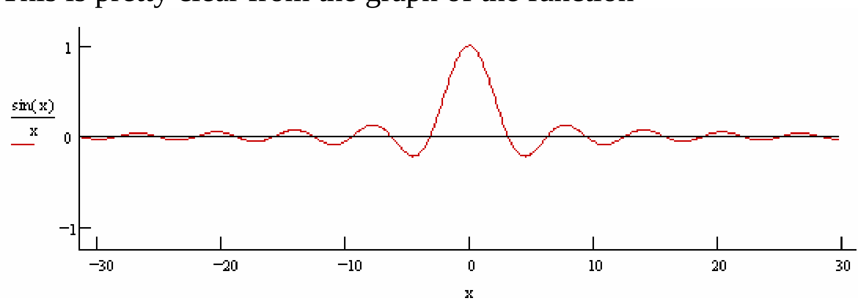
  So $\lim\limits_{x\to 2^-}\dfrac{x^2-4}{|x-2|} = \lim\limits_{x\to 2^-}\dfrac{(x+2)(x-2)}{-(x-2)}$

  $= \lim\limits_{x\to 2^-}\dfrac{x+2}{-1} = -4$

  Therefore, $\lim\limits_{x\to 2}\dfrac{x^2-4}{|x-2|}$ D.N.E.

__Limits of Trig Functions:__

- A famous limit is given by $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$.
- This is pretty clear from the graph of the function



- But it can also be found with the squeeze theorem, because near $x = 0$, $\cos x < \dfrac{\sin x}{x} < 1$. And both $\lim\limits_{x\to 0}\cos x = 1$ and $\lim\limits_{x\to 0} 1 = 1$.

- *Example. Evaluate* $\lim_{x\to 0}\frac{\sin(2x)}{x}$

$$\lim_{x\to 0}\frac{2\sin(2x)}{2x} = 2\lim_{2x\to 0}\frac{\sin(2x)}{2x}$$
$$= 2(1)$$
$$= 2$$

- *Example. Evaluate* $\lim_{x\to 0}\frac{\cos x - 1}{\sin x}$

$$\lim_{x\to 0}\frac{\cos x - 1}{\sin x} = \lim_{x\to 0}\frac{\cos x - 1}{\sin x}\frac{\cos x + 1}{\cos x + 1}$$
$$= \lim_{x\to 0}\frac{\cos^2 x - 1}{\sin x(\cos x + 1)}$$
$$= \lim_{x\to 0}\frac{-\sin^2 x}{\sin x(\cos x + 1)}$$
$$= \lim_{x\to 0}\frac{-\sin x}{\cos x + 1} = 0$$