Section 1.3 – Limit of a Function

A Beginning for Limits:

- We want to approximate $f(3)$ when $f(x) = 4x - 5$. I realize we can actually find this by plugging 3 into $f(3) = 4(3) - 5 = 7$. But let's just see what is happening
- Maybe because we are so far away from 3 on each side it is hard to tell what is happening…

x	f(x)
0	-5
1	-1
2	3
3	?
4	11
5	15
6	19

- So let's zoom in to get a better approximation for $f(3)$

x	f(x)
2.9	6.6
2.99	6.96
2.999	6.996
3	?
3.001	7.004
3.01	7.04
3.1	7.4

- So we can see that as x gets closer to 3 from each side $f(x)$ is getting close to 7. The closer we zoom in, the more easy it will be to tell we are approaching 7.

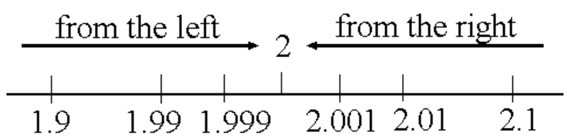
Getting Used to Terminology:

- In the above example, we are getting closer and closer (from the left and right) to the value of 3, and in turn we are getting closer and closer to approximating $f(3)$.
- When doing limits, it doesn't matter that we don't quite get there, just as long as that is where we will end up as we continue to move closer from each side.

- Let's look at another example with $f(x) = 2x + 1$.

  This is a linear function. It has no strange behavior, it is just a line.
  We can evaluate $f(2)$ pretty easily by plugging in $x = 2$ into the equation.
  Q: What is $f(2)$?
  A: ____.

  From a limit perspective, let's look at the points 'near' 2…



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2.001	5.002
2.01	5.02
2.1	5.2

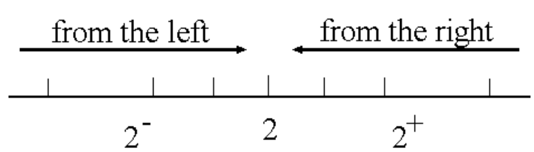
  Notice that we are getting closer to 2 from each side (from the left and from the right)

  Q: What is the limit of $f(x)$ as x tends towards 2 from the right and left?
  A: ____.

  Q: How does this compare with the value we found above for $f(2)$?
  A: ____.

One Sided Limits:

- If we are approaching a value **from the left**, we indicate this with a minus sign as a subscript next to the value. If we are approaching a value **from the right**, we indicate this with a plus sign subscript.



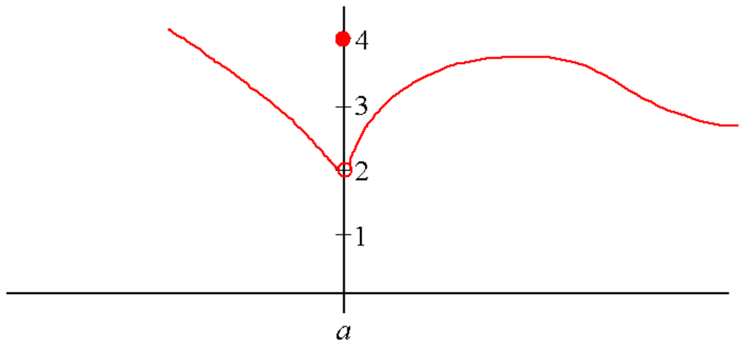
- This has nothing to do with the sign of the value, but only which side you approach from.
- Q: What would $x \to -1^+$ indicate?
  A: ____.

- If a plus/minus sign is omitted (meaning it is not there) it means you are approaching both from the **right AND the left**.
- If we want to find the limit of a function $f(x)$ as x approaches some value a, we indicate this symbolically with $\lim_{x \to a} f(x)$.

- Q: How would you notate the limit of $f(x)$ as x approaches a from the right? From the left?

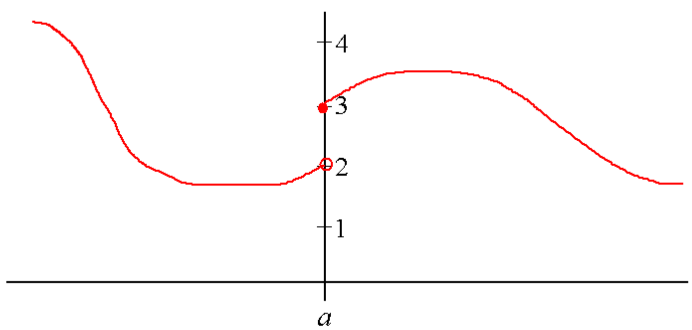
  A: ____.

Looking at Limits Graphically:



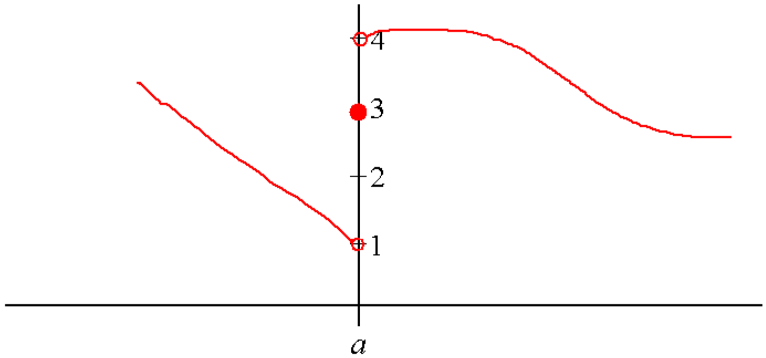
- Q: Find the following for the graph above: $f(a)$, $\lim_{x \to a^+} f(x)$, $\lim_{x \to a^-} f(x)$, $\lim_{x \to a} f(x)$

  A: ____.



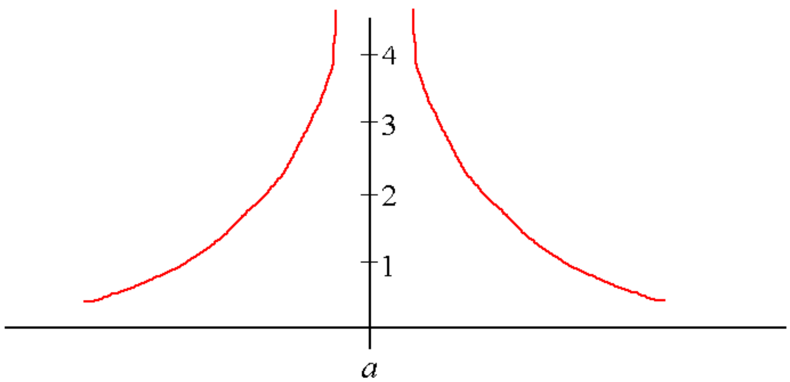
- Q: Find the following for the graph above: $f(a)$, $\lim_{x \to a^+} f(x)$, $\lim_{x \to a^-} f(x)$, $\lim_{x \to a} f(x)$

  A: ____.



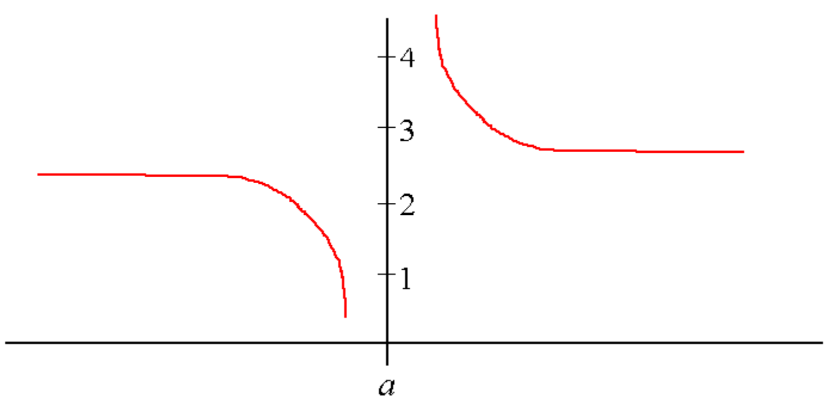
- Q: Find the following for the graph above: $f(a)$, $\lim_{x\to a^+} f(x)$, $\lim_{x\to a^-} f(x)$, $\lim_{x\to a} f(x)$

  A: ____.



- Q: Find the following for the graph above: $f(a)$, $\lim_{x\to a^+} f(x)$, $\lim_{x\to a^-} f(x)$, $\lim_{x\to a} f(x)$

  A: ____.

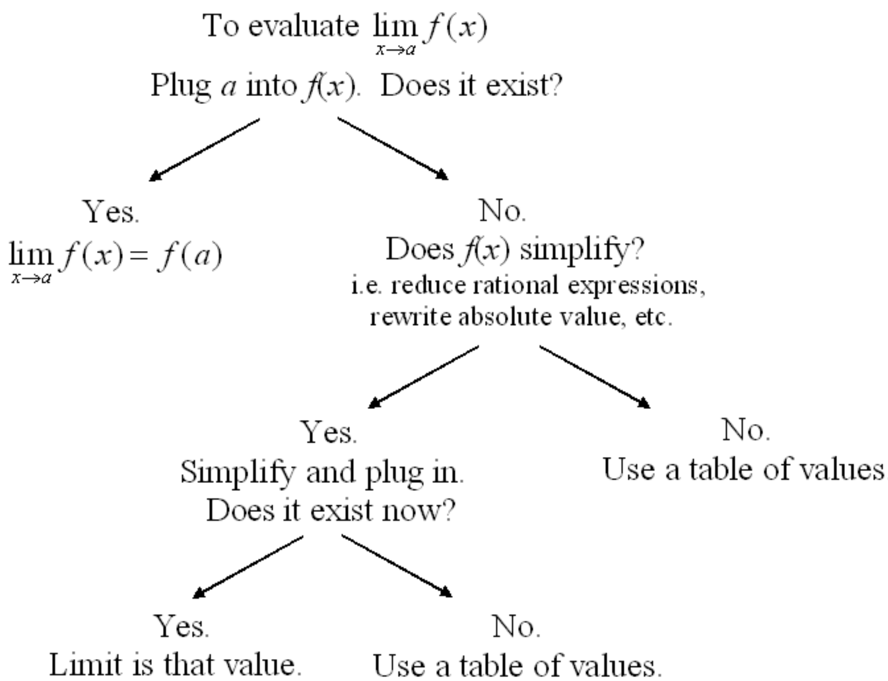


- Q: Find the following for the graph above: $f(a)$, $\lim_{x\to a^+} f(x)$, $\lim_{x\to a^-} f(x)$, $\lim_{x\to a} f(x)$

  A: ____.

"Crude" Method for Finding Limits of Functions without a Graph:

- If you are not given the graph of a function, don't just blindly make a table of values, follow the outline below and you will usually have success. Also, be weary of using your graphing calculator, as it is often wrong when graphing



- *Example, find* $\lim_{x\to 4} \frac{1}{x}$

- *Example, find* $\lim_{x\to 2} \frac{x^2-4}{x-2}$

- *Example, find* $\lim_{x \to 0} \frac{\sqrt{x+4}-2}{x}$

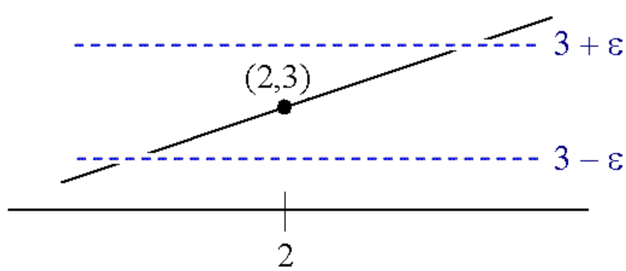
- *Example, find* $\lim_{x \to -1^+} \frac{x-3}{x^2(x+1)}$

The Formal Definition of a Limit:

- First realize there is a difference between **finding** a limit, and **proving** a limit is a certain value.
- What we just completed in this section is **finding** the limit. What we are about to do now is **proving** what the limit is.
- We will begin by working our way to the formal definition of a limit.

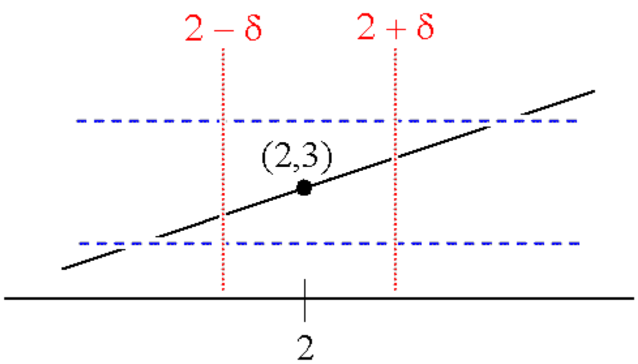
Translating the Pictures to a Definition:

- Let's look at a specific function, $f(x) = x + 1$.
- This is a linear function, so it has a limit for every domain value, but let's pick $a = 2$.
- Now we've already learned that $\lim_{x \to 2} x+1=3$, but now we want to **formally prove** it is true.
- The horizontal lines we pick arbitrarily are centered about the limit L, in this case L=3



We require our function to be inside these lines, that is, $f(x) < 3+\varepsilon$ and $f(x) > 3-\varepsilon$.
We can say this in one step, that is, $|f(x)-3| < \varepsilon$.
So a way to specify the horizontal lines in general would be: $|f(x)-L| < \varepsilon$ for any $\varepsilon > 0$.

- Now for the vertical lines. Remember, our function has to go in and out the sides.



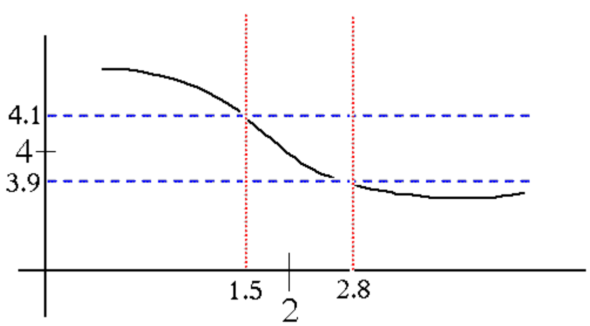
We found two red lines that will work for any *x* between them, that is $x > 2-\delta$ and $x < 2+\delta$.
We can say this in one step with $|x-2| < \delta$.
So a way to specify the vertical lines in general would be to say $|x-a| < \delta$.

- So remember our game… for any $\varepsilon > 0$ that is chosen, we have to *find* a $\delta > 0$ so that as long as $|x-a| < \delta$ we satisfy $|f(x)-L| < \varepsilon$

- There you have it! The formal definition of a limit…
The **limit of *f*(*x*) as *x* approaches *a* is *L***, i.e. $\lim_{x\to a} f(x) = L$ if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x)-L| < \varepsilon$ whenever $|x-a| < \delta$.

Another Small Step (that the book uses):

- *Example. Use the graph to find delta so* $|f(x)-4| < 0.1$ whenever $|x-2| < \delta$.



In the box you are looking at, it may be hard to see, but the function is touching the corners.
Plus, it is not symmetric about *x* = 2.
Remember it needs to be the same distance on the right and left of 2
The distance from 1.5 to 2 is 0.5, and the distance from 2 to 2.8 is 0.8.
We need to take the smaller of these, 0.5
So *any* positive value of delta less than (or equal to) 0.5 will work. i.e. 0.5, 0.49, 0.4

- *Example. For* $f(x)=x^2$*, find a number delta >0 such that* $\left|x^2-4\right|<0.5$ *whenever* $|x-2|<\delta$.

  In this example, we are trying to "prove" $\lim_{x\to 2} x^2 = 4$ with a specific epsilon = 0.5

  For this example, L = 4 and a = 2.

  $\left|x^2-4\right|<0.5 \Rightarrow -0.5<x^2-4<0.5, \quad x^2>3.5$ and $x^2<4.5$

  So the top line of the box is at 4.5, and the bottom line of our box is 3.5.

  You need to find out what input gives these two outputs. i.e. $f^{-1}(3.5)$ and $f^{-1}(4.5)$

  $f^{-1}(x)=\sqrt{x}$ for $x > 0$, so $f^{-1}(3.5)\approx 1.87$ and $f^{-1}(4.5)\approx 2.12$

  We pick the smallest of these distances from 2 (recall a = 2), so delta = 0.12 (or less).

<u>A Practical Example:</u>

- Prove that $\lim_{x\to 3}(2x-5)=1$.
- Here, $f(x) = 2x - 5$. $a = 3$. $L = 1$.
- Let $\varepsilon>0$ be given and assume that $\left|x-3\right|<\delta$ for some $\delta>0$.
- Remember, it is our job to find that delta!
- $\left|f(x)-L\right|=\left|2x-5-1\right|=\left|2x-6\right|=2\left|x-3\right|$
- And we are assuming that $\left|x-3\right|<\delta$, so $\left|f(x)-L\right|=2\left|x-3\right|<2\delta$.
- So we want to force $\left|f(x)-L\right|<\varepsilon$, so we take $2\delta<\varepsilon \Rightarrow \delta<\dfrac{\varepsilon}{2}$.
- So for $\varepsilon>0$ there is a $\delta>0$ $\left(\delta<\dfrac{\varepsilon}{2}\right)$ such that $\left|f(x)-1\right|<\varepsilon$ whenever $\left|x-3\right|<\delta$.

<u>Supplemental Exercises:</u>

1. Prove $\lim_{x\to 4}(-x+3)=-1$
2. Prove $\lim_{x\to -3}(2x+1)=-5$
3. Prove $\lim_{x\to 0}(6x+3)=3$
4. Prove $\lim_{x\to -2}(-x+3)=5$