Section 1.3 – Limit of a Function

A Beginning for Limits:

- We want to approximate the value for $\sqrt{2}$.
- Granted, we could find the limit as x approaches 2 for $f(x) = \sqrt{x}$.

x	f(x)
1.5	1.22474
1.9	1.3784
1.99	1.41067
1.999	1.41386
2	?
2.001	1.41457
2.01	1.41774
2.1	1.44914
2.5	1.58114

- So we can see that the value for $\sqrt{2}$ is between 1.41386 and 1.41457.
- Note that we can get a better approximation as we move closer to 2 on each side.

x	f(x)
1.9999	1.41418
1.99999	1.41421
2	?
2.00001	1.41422
2.0001	1.41425

- So we can see that the value for $\sqrt{2}$ is between 1.41421 and 1.41422.

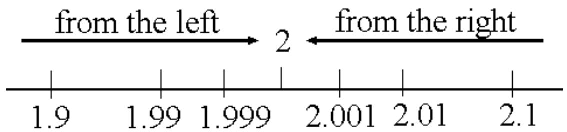
Getting Used to Terminology:

- In the above example, we are getting closer and closer (from the left and right) to the value of 2, and we are approximating the value of $\sqrt{2}$.
- When doing limits, it doesn't matter that we don't quite get there, just as long as that is where we will end up as we continue to move closer from each side.

- Let's look at another example with $f(x) = 2x + 1$.

  This is a linear function. It has no strange behavior, it is just a line.
  We can evaluate $f(2)$ pretty easily by plugging in $x = 2$ into the equation.
  Q: What is $f(2)$?
  A: $f(2) = 2(2)+1 = 5$

  From a limit perspective, let's look at the points 'near' 2…



--	--
2.001	5.002
2.01	5.02
2.1	5.2

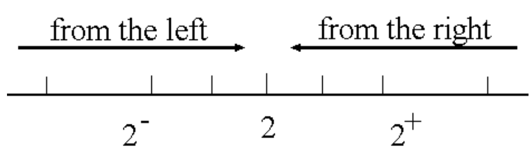
  Notice that we are getting closer to 2 from each side (from the left and from the right)

  Q: What is the limit of $f(x)$ as x tends towards 2 from the right and left?
  A: It seems it is tending to 5.

  Q: How does this compare with the value we found above for $f(2)$?
  A: It is the same.

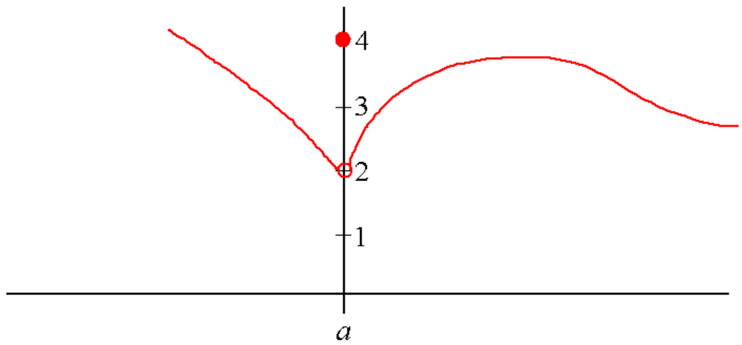
<u>One Sided Limits:</u>

- If we are approaching a value **from the left**, we indicate this with a minus sign as a subscript next to the value. If we are approaching a value **from the right**, we indicate this with a plus sign subscript.



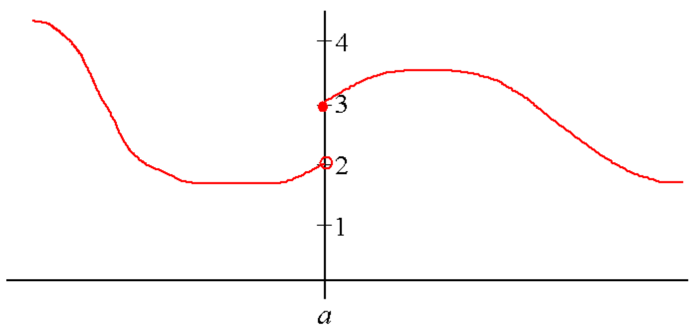
- This has nothing to do with the sign of the value, but only which side you approach from.
- Q: What would $x \to -1^+$ indicate?
  A: Approaching negative 1 from the right side.
- If a plus/minus sign is omitted (meaning it is not there) it means you are approaching both from the **right AND the left**.
- If we want to find the limit of a function $f(x)$ as x approaches some value a, we indicate this symbolically with $\lim_{x \to a} f(x)$.
- Q: How would you notate the limit of $f(x)$ as x approaches a from the right? From the left?
  A: Limit from the right would be $\lim_{x \to a^+} f(x)$. Limit from the left would be $\lim_{x \to a^-} f(x)$

Looking at Limits Graphically:



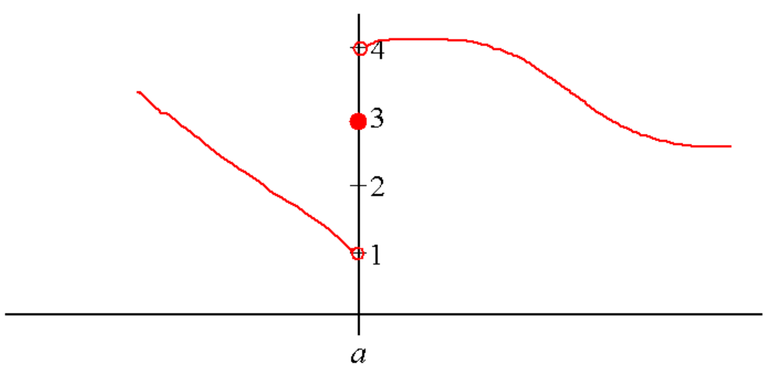
- Q: Find the following for the graph above: $f(a)$, $\lim_{x\to a^+} f(x)$, $\lim_{x\to a^-} f(x)$, $\lim_{x\to a} f(x)$

  A: $f(a)=4$ $\lim_{x\to a^+} f(x)=2$ $\lim_{x\to a^-} f(x)=2$ $\lim_{x\to a} f(x)=2$



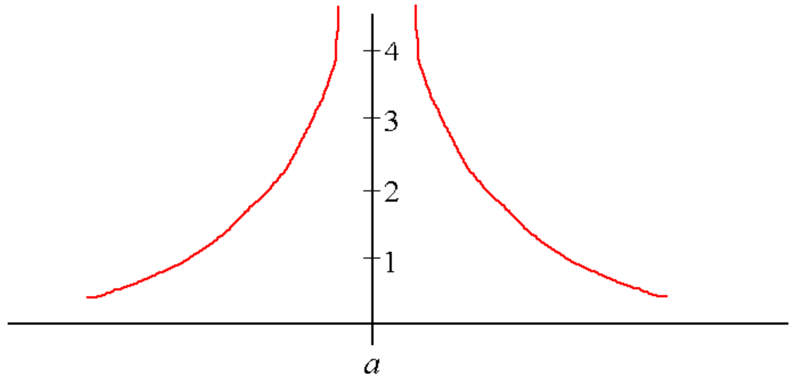
- Q: Find the following for the graph above: $f(a)$, $\lim_{x\to a^+} f(x)$, $\lim_{x\to a^-} f(x)$, $\lim_{x\to a} f(x)$

  A: $f(a)=3$ $\lim_{x\to a^+} f(x)=3$ $\lim_{x\to a^-} f(x)=2$ $\lim_{x\to a} f(x)=dne$



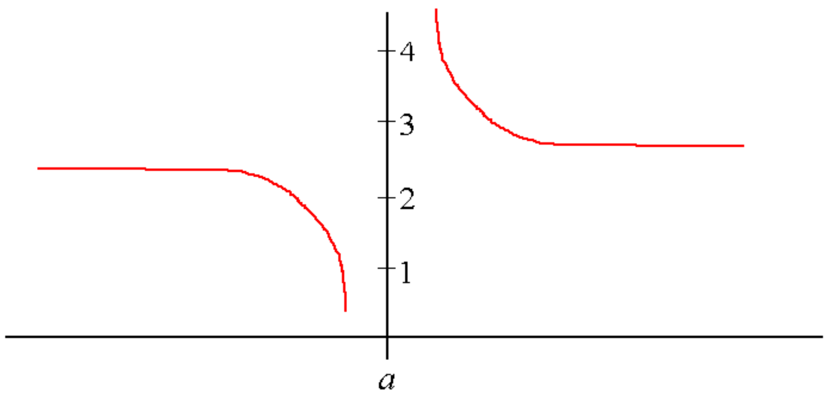
- Q: Find the following for the graph above: $f(a)$, $\lim_{x\to a^+} f(x)$, $\lim_{x\to a^-} f(x)$, $\lim_{x\to a} f(x)$

  A: $f(a)=3$ $\lim_{x\to a^+} f(x)=4$ $\lim_{x\to a^-} f(x)=1$ $\lim_{x\to a} f(x)=dne$



- Q: Find the following for the graph above: $f(a)$, $\lim_{x \to a^+} f(x)$, $\lim_{x \to a^-} f(x)$, $\lim_{x \to a} f(x)$

  A: $f(a) = dne$ $\lim_{x \to a^+} f(x) = \infty$ $\lim_{x \to a^-} f(x) = \infty$ $\lim_{x \to a} f(x) = \infty$

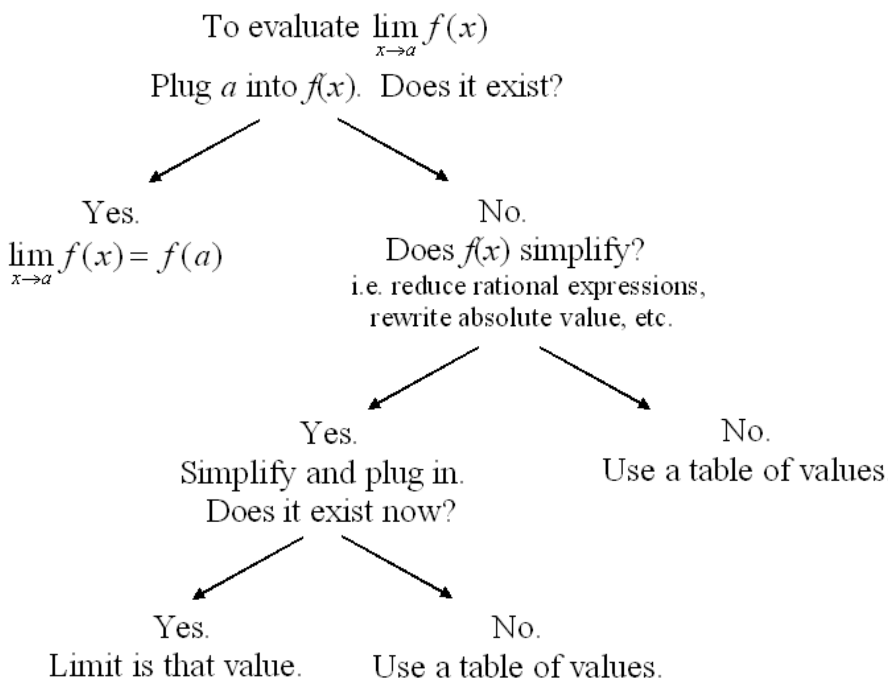


- Q: Find the following for the graph above: $f(a)$, $\lim_{x \to a^+} f(x)$, $\lim_{x \to a^-} f(x)$, $\lim_{x \to a} f(x)$

  A: $f(a) = dne$ $\lim_{x \to a^+} f(x) = \infty$ $\lim_{x \to a^-} f(x) = -\infty$ $\lim_{x \to a} f(x) = dne$

<u>"Crude" Method for Finding Limits of Functions without a Graph:</u>

- If you are not given the graph of a function, don't just blindly make a table of values, follow the outline below and you will usually have success. Also, be weary of using your graphing calculator, as it is often wrong when graphing



- *Example, find* $\lim_{x \to 4} \frac{1}{x}$

$$\lim_{x \to 4} \frac{1}{x} = \frac{1}{4}$$

- *Example, find* $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x-2}$$
$$= \lim_{x \to 2} \frac{x+2}{1}$$
$$= 4$$

- *Example, find* $\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}$

$$\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}=\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}\left(\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}\right)$$
$$=\lim_{x\to 0}\frac{(x+4)-4}{x(\sqrt{x+4}+2)}$$
$$=\lim_{x\to 0}\frac{1}{(\sqrt{x+4}+2)}$$
$$=\frac{1}{4}$$

- *Example, find* $\lim_{x\to -1^+}\frac{x-3}{x^2(x+1)}$

$$\lim_{x\to -1^+}\frac{x-3}{x^2(x+1)}=-\infty$$

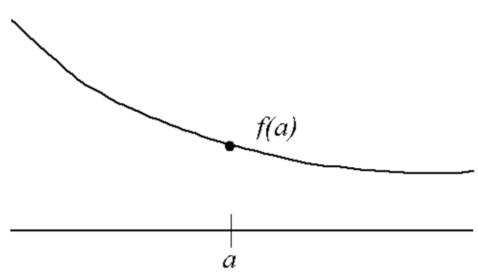
NOTE: This problem cannot be simplified, it will 'blow up' as x tends to 1. Your only job is to figure out if it is negative or positive infinity.

The Formal Definition of a Limit:

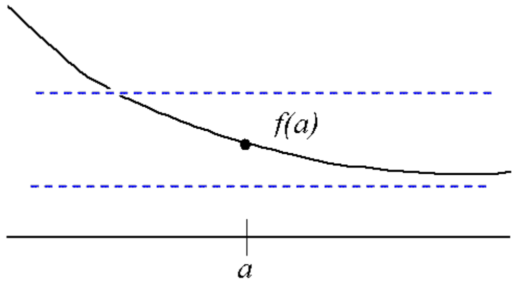
- First realize there is a difference between **finding** a limit, and **proving** a limit is a certain value.
- What we just completed in this section is **finding** the limit. What we are about to do now is **proving** what the limit is.
- We will begin by working our way to the formal definition of a limit.

A First Look, When Things Go Right:

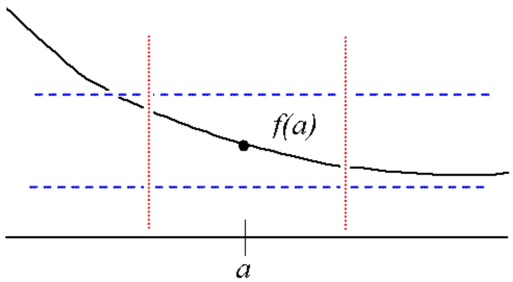
- Q: From what we saw from the last section, does the limit of $f(x)$ exist at $x = a$ for the graph below?
A: Yes it does, because the limit from the left and right are the same.



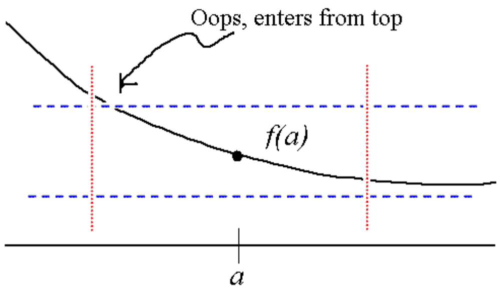
- Let's pick two horizontal lines that are the same distance from $f(a)$, which is the limit, and plot them on our graph. We can make them as close or as far from $f(a)$ as we want, it is our choice



- Now we have to make a rectangle with two vertical lines that are the same distance from a, but the trick is that we have to make sure our function only enters and exits our rectangle from the sides, not the top or bottom.

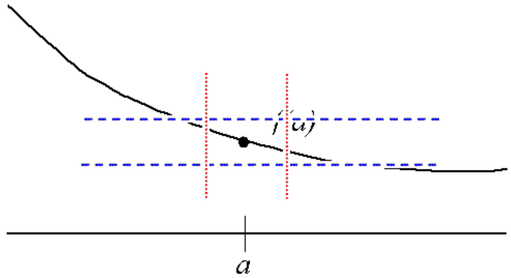


CORRECT



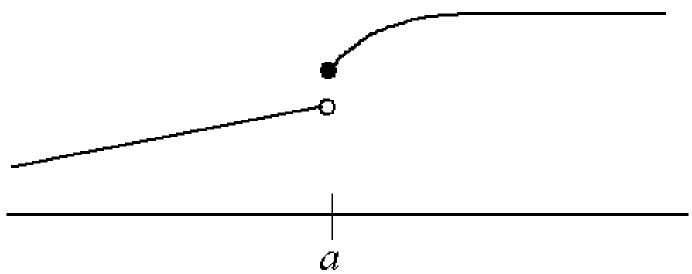
INCORRECT

- In this example, no matter how small we make our horizontal lines, we can find vertical lines to make it work.



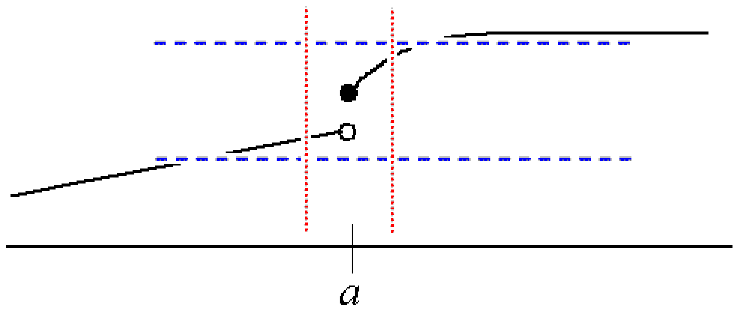
A First Look, When Things Go Wrong:

- Q: From what we saw before, does the limit of $f(x)$ exist at $x = a$ for the graph below?

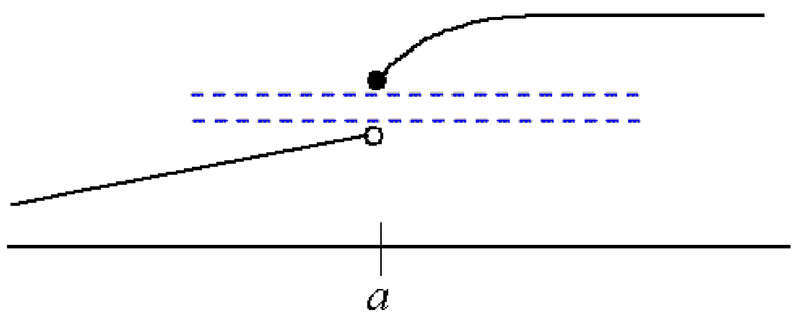


A: No, because it tends to two different values from the left and right.

- Well, it looks like for the horizontal lines I've chosen, I can get a box around them.



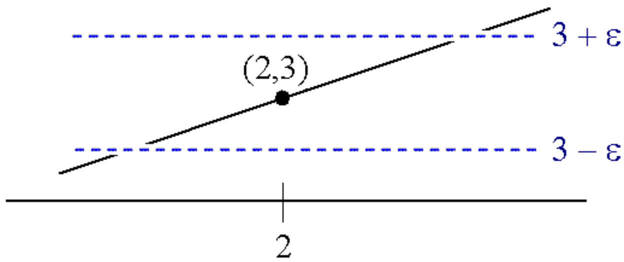
- But remember, I can pick *any* horizontal lines I want and it still has to work.



- This is bad news. There is no way I can even get my function inside the box at all, let alone enter and exit from the sides.
- The limit here does not exist at *a*.

<u>Translating the Pictures to a Definition:</u>

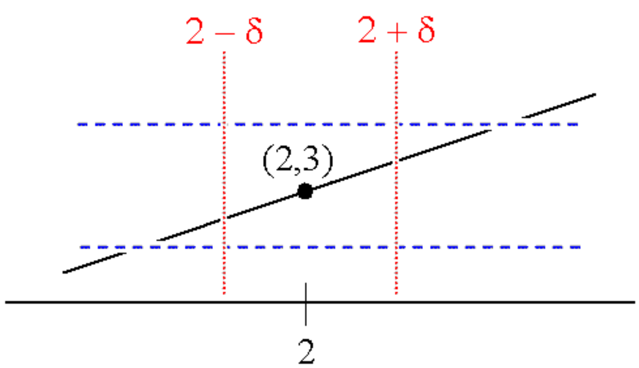
- Let's look at a specific function, $f(x) = x + 1$.
- This is a linear function, so it has a limit for every domain value, but let's pick $a = 2$.
- Now we've already learned that $\lim_{x \to 2} x + 1 = 3$, but now we want to **formally prove** it is true.
- The horizontal lines we pick arbitrarily are centered about the limit L, in this case $L=3$



We require our function to be inside these lines, that is, $f(x) < 3 + \varepsilon$ and $f(x) > 3 - \varepsilon$.
We can say this in one step, that is, $|f(x) - 3| < \varepsilon$.

So a way to specify the horizontal lines in general would be: $|f(x) - L| < \varepsilon$ for any $\varepsilon > 0$.

- Now for the vertical lines. Remember, our function has to go in and out the sides.



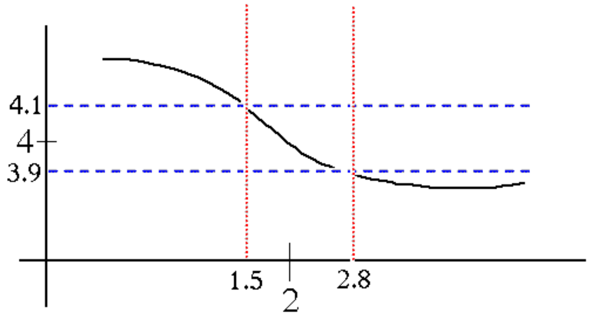
We found two red lines that will work for any x between them, that is $x > 2 - \delta$ and $x < 2 + \delta$.
We can say this in one step with $|x - 2| < \delta$.

So a way to specify the vertical lines in general would be to say $|x - a| < \delta$.

- So remember our game… for any $\varepsilon > 0$ that is chosen, we have to *find* a $\delta > 0$ so that as long as $|x - a| < \delta$ we satisfy $|f(x) - L| < \varepsilon$
- There you have it! The formal definition of a limit…
The **limit of *f*(*x*) as *x* approaches *a* is *L***, i.e. $\lim_{x \to a} f(x) = L$ if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$.

Another Small Step (that the book uses):

- *Example. Use the graph to find delta so* $|f(x)-4|<0.1$ whenever $|x-2|<\delta$.



In the box you are looking at, it may be hard to see, but the function is touching the corners.
Plus, it is not symmetric about $x = 2$.
Remember it needs to be the same distance on the right and left of 2
The distance from 1.5 to 2 is 0.5, and the distance from 2 to 2.8 is 0.8.
We need to take the smaller of these.
So *any* positive value of delta less than (or equal to) 0.5 will work. i.e. 0.5, 0.49, 0.4

- *Example. For* $f(x)=x^2$*, find a number delta >0 such that* $|x^2-4|<0.5$ *whenever* $|x-2|<\delta$.

In this example, we are trying to "prove" $\lim_{x\to 2} x^2 = 4$ with a specific epsilon = 0.5

$|x^2-4|<0.5 \;\Rightarrow x^2>3.5$ and $x^2<4.5$

These are the top and bottom lines of your box.
You need to find out what input gives these two outputs. i.e. $f^{-1}(3.5)$ and $f^{-1}(4.5)$

$f^{-1}(x)=\sqrt{x}$, so $f^{-1}(3.5)\approx 1.87$ and $f^{-1}(4.5)\approx 2.12$

We pick the smallest of these distances from 2, so delta = 0.12 (or less)

A Practical Example:

- Prove that $\lim_{x\to 3}(2x-5)=1$.
- Here, $f(x) = 2x - 5$. $a = 3$. $L = 1$.
- Let $\varepsilon>0$ be given and assume that $|x-3|<\delta$ for some $\delta>0$.
- Remember, it is our job to find that delta!
- $|f(x)-L|=|2x-5-1|=|2x-6|=2|x-3|$
- And we are assuming that $|x-3|<\delta$, so $|f(x)-L|=2|x-3|<2\delta$.
- So we want to force $|f(x)-L|<\varepsilon$, so we take $2\delta<\varepsilon \;\Rightarrow \delta<\dfrac{\varepsilon}{2}$.
- So for $\varepsilon>0$ there is a $\delta>0$ $\left(\delta<\dfrac{\varepsilon}{2}\right)$ such that $|f(x)-1|<\varepsilon$ whenever $|x-3|<\delta$.

Supplemental Exercises:

1. Prove $\lim_{x\to 4}(-x+3)=-1$
2. Prove $\lim_{x\to -3}(2x+1)=-5$
3. Prove $\lim_{x\to 0}(6x+3)=3$
4. Prove $\lim_{x\to -2}(-x+3)=5$