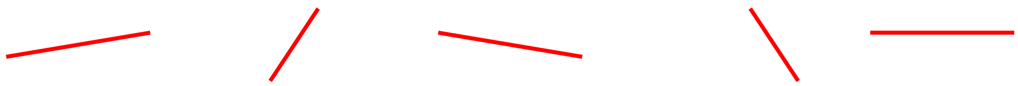
Section 1.2 – A Catalog of Essential Functions

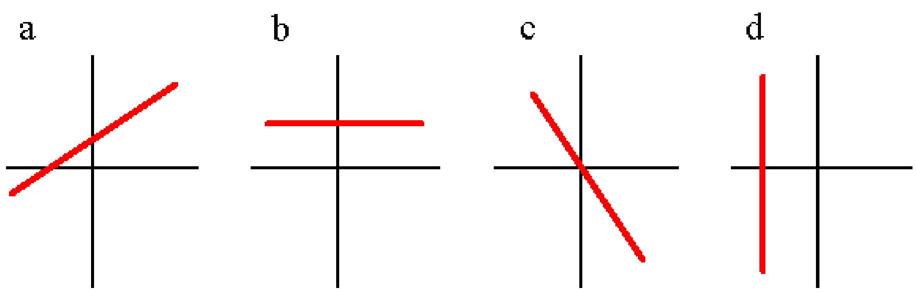
Linear Models:

- All linear equations have the form **$y = mx + b$**.
- The letter m is the **slope** of the line, $\frac{\text{rise}}{\text{run}}$ or $\frac{\text{change in horizontal}}{\text{change in vertical}}$. It can be positive, negative or zero. It can also be very large or very small.
- Q: What would the line look like in each one of these cases?
  small/positive large/positive small/negative large/negative zero
  A:



- The letters x and y are **variables**, meaning they vary or change along the line. At least one of them must be present in the equation. Together they represent the **ordered pair** (x, y)
- The letter b represents the **y (or horizontal) intercept**, this is where the line crosses the horizontal axis.
- Q: What are the restrictions on this value?
  A: The y intercept can be positive, negative, small, large or zero.
- *Example: Below are three different graphs along with 4 different equations. Match them.*

  Equations: 1. $y = -0.5x$ 2. $y = 3$ 3. $x = -3$ 4. $y = 2x + 5$



Equation 1 has a y-intercept of 0, and a small negative slope. It matches with 'c'

Equation 2 has a y-intercept of 3 and a zero slope. It matches with 'b'

Equation 3 has no slope (since y is missing) which means it is a vertical line. 'd'

Equation 4 has a y-intercept of 5, and a positive slope. It matches with 'a'.

Polynomials:

- Recall: a **polynomial in one variable** is an expression of the form:

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

  where the a_i's are real number **coefficients**.

- For nonzero a_n, the expression is said to be of *n*th degree (the highest power is *n*), the **leading term** is $a_n x^n$ and the **leading coefficient** is a_n.

- Examples of polynomials that are common

Degree	Name	Form
0	Constant	$f(x) = c$
1	Linear	$f(x) = mx + b$
2	Quadratic	$f(x) = ax^2 + bx + c$
3	Cubic	$f(x) = ax^3 + bx^2 + cx + d$

Power Functions:

- A **power function** is of the form $f(x) = x^a$, where *a* is any constant.

- Q: What is the difference between a power function and a polynomial?
  A: In a polynomial, the exponent must be a whole number (0,1,2,3…). In a power function, it is allowed to be anything. If *a* happens to be whole, it is a polynomial (with one term) of degree *a*.

- Q: Is a polynomial a power function?
  A: It can be, but is 'usually' not because polynomials typically have more than one term.
  i.e. $f(x) = x^2$ is a polynomial and a power function.

- If the value of *a* is a fraction, the power function is also called a **root function**.

- *Example.* $f(x) = x^{1/2} = \sqrt{x}$

  This is a power function where a = 1/2

- If the value of *a* is negative, it is the **reciprocal function**.

- *Example.* $f(x) = x^{-1} = \dfrac{1}{x}$

  This is a power function where a = –1

Rational Functions:

- A rational function is a ratio of two polynomials $\dfrac{p(x)}{q(x)}$.

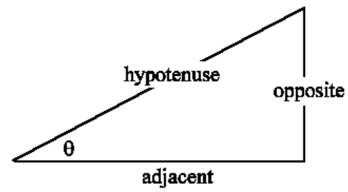
- If the rational function has a root, it would be when $p(c) = 0$ (so long as $q(c)$ isn't zero).

- The rational function will be undefined (at *c*) whenever $q(c) = 0$.

Trigonometric Functions:

- The six trig functions are defined in terms of right triangles (see figure). The two "most important" being sine and cosine.

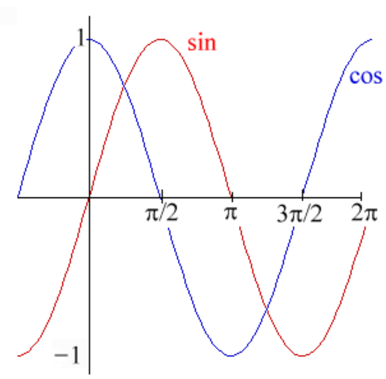
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$



- All other trig functions can be defined in terms of sine and cosine, so remember the definitions above, and the relationships between them and the others…

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

- To evaluate trig functions, remember some fundamentals.
- First, remember the basic shape of sin(x) and cos(x), along with when they are 0, and ± 1.



- From the figure you can see they are periodic, both with a period of 2π.
- Q: When is sine equal to zero?
  A: All multiples of pi $(0, \pm\pi, \pm 2\pi, \ldots)$
- Q: When is cosine equal to zero?
  A: All odd multiples of pi over two $\left(\frac{\pm\pi}{2}, \frac{\pm 3\pi}{2}, \frac{\pm 5\pi}{2} \ldots\right)$

Exponential Functions:

- $y = kb^t$
- Variables are *y* and *t*. *y* is the dependent variable and *t* is the independent variable.
- *b* is the **base** ($b > 0,\ b \neq 1$).
- *k* is the **initial quantity** (when *t* = 0).
- A commonly used base is *e* = 2.7182818284…
- *Rules for Exponentials:*

  $$b^x b^y = b^{x+y}$$

  $$(b^x)^y = b^{xy}$$

  $$b^{-x} = \frac{1}{b^x}$$

Log Functions:

- We want to undo the exponential function $b^y = x$
- This is true if and only if $y = \log_b x$
  y is the exponent, *b* is the base, and *x* is the argument.
- So the log function is the **inverse** of the exponential function.
- Q: What are our conditions on *x*, and *b*?
  A: *x* and *b* must be greater than zero.
- *Example. Change to exponential form to solve* $\log_{10} \frac{1}{100} = ?$

  $$10^{?} = \frac{1}{100} = 10^{-2}$$

  $$? = -2$$

- *Example. Convert* $e^{-t} = 4000$ *to log.*

  $$\log_e 4000 = -t$$

  $$t = -\log_e 4000$$

- **Special Log Bases:**
  Log base *e* is natural log (written ln)
  Log base 10 is common log (written log)
- These will be the only two on your calculator. So if you need to calculate say, $\log_4 2$, you have to use the change of base formula $\log_b M = \frac{\log_a M}{\log_a b}$.

- *Example, use the change of base formula to evaluate* $\log_4 2$

$$\log_4 2 = \frac{\ln 2}{\ln 4} = \frac{\log 2}{\log 4} = 0.5$$

Algebraic and Transcendental Functions

- An **algebraic function** is constructed with algebraic operations (addition, subtraction, multiplication and division). A **transcendental function** is anything else.
- Q: Classify all the functions we have looked at so far as algebraic or transcendental… Log, Exponential, Trig, Rational, Power, Polynomials
  A: Transcendental: Log, Exponential, Trig. Algebraic: Rational, Power, Polynomial

Transformations of Functions

- NOTE: There are ways of transforming functions by shifting them left, right, up or down. Or by stretching them, or shrinking them. Or by reflecting them… BUT… once we learn the *actual* tools for graphing (after differentiation) it usually more straightforward to graph what you are given.
- So we will not go over transformations in class, but I expect you will look over the material and bring me any questions you might have.
- Some functions (like trigs, exponentials and logs) are, admittedly, easier to do with transformations.

Composition of Functions:

- Sometimes it is helpful to break things into different parts, and recognize how they fit together.
- A composite function works to accomplish this goal, because it identifies functions inside functions.
- For example, we can think of the function $f(x) = (x+1)^2$ as the composite of two functions, one function is the inside piece $in(x) = x+1$ and the other is the outside piece $out(x) = x^2$. So we see that $f(x) = out(in(x))$.
- The **composite function** $f \circ g(x)$, the **composition** of *f* and *g*, is defined as $(f \circ g)(x) = f(g(x))$. The domain of *g* is *x* and the domain of *f* is *g*(*x*).
- *Example. Find* $(g \circ h)(1/2)$ *if* $g(x) = x^2 - 2x - 6$ *and* $h(x) = x^3$

  We want to find *g*(*h*(1/2)).

  1/2 is in the domain of *h*, and *h*(1/2) = 1/8.

  So, 1/8 is in the domain of *g*, and

$$g(1/8) = (1/8)^2 - 2(1/8) - 6 = \frac{1}{64} - \frac{2(8)}{64} - \frac{6(64)}{64} = \frac{-399}{64}$$

  So *g*(*h*(1/2)) = *g* (1/8) = –399/64.

- *Example. Find the composites and domains if* $f(x)=\dfrac{6}{x}$ *and* $g(x)=\dfrac{1}{2x+1}$

  $(f\circ g)(x)=\dfrac{6}{1/(2x+1)}=6(2x+1)$

  For the domain of *g*: $2x+1\neq 0 \Rightarrow x\neq -1/2$.
  For the domain of *f*: $x\neq 0$ which means that $g(x)\neq 0$, which it never is anyway.
  So the domain of $f\circ g$ is all *x* except *x* = –1/2.

  $(g\circ f)(x)=\dfrac{1}{2\frac{6}{x}+1}=\dfrac{x}{12+x}$

  For the domain of *f*: $x\neq 0$.
  For the domain of *g*, $2x+1\neq 0 \Rightarrow x\neq -1/2$ which means $f(x)\neq -1/2$ so $\dfrac{6}{x}\neq\dfrac{-1}{2}\Rightarrow x\neq -12$.
  So the domain of $g\circ f$ is all *x* except 0 and –12.