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Application of optimal control theory in marketing: What is the optimal number of choices on a shopping website?

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Abstract: There have been some recent interests in making websites responsive to the dynamic behaviour of users. We present a method for applying the steps of optimal control theory to internet shopping platforms. The information rate is a state variable to maximise and the rate of change in number of options available to the consumer is our control variable to minimise. We apply optimal control theory to identify the optimal number of options. We specify and solve the necessary and sufficient conditions for the optimal control for a class of objective functions, which gives a nonlinear two-point boundary value problem.

Keywords: optimal control; feedback control; marketing science; two-point boundary value problem; dynamic e-commerce.

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1 Introduction

The internet presents a unique medium through which firms can identify information about consumers and then react to it. This two-way interaction has been the subject of much marketing, information systems, and computing science research. The interest of such research is often placed on how users and their interactions can be simulated, in order to assess their goals and tie those goals into their actual behaviours on internet sites (Chi et al., 2001; Park and Fader, 2004; Bucklin and Sismeiro, 2003). In our research, we ask how the internet site itself can adapt to the users. Many of the large internet vendors, such as Amazon.com, have demonstrated the viability of gathering user-specific information and using it to present more tailored information to consumers. To that end, there has been some recent interest in making websites responsive to the dynamic behaviour of users (Chi et al., 2000; Perkowitz and Etzioni, 1999, 2000). However, there has not been any framework that uses the mechanisms developed in the area of feedback control theory. In this paper, we first present a survey of the various ways web sites are dynamically changed based on the direct or indirect input from the users. Then, we follow the description of Krishen and Nakamoto (2009) of how the steps of feedback control design can be applied to some types of design for websites. As an example of using optimal control theory for web design, we may consider the problem of deciding how many choices to provide to a user in a choice set. More specifically, in a computer shopping platform, the main question is how many different computer configurations should be shown to the customers at each stage. Too many choices could result in an information overload condition, in which the user is unable and unwilling to sort through. This leads to a low information processing rate. Not enough choices would also cause the user to take too much time to come to his final choice, because despite the fact that for each pop-up window or page shown to customer the processing time is low, it takes more time to cover all available choices. So, the overall information processing time will be low. The optimal number of choices, though, is not a constant or fixed number, but instead varies as a function of time and individual. Information rate, defined as the number of choices presented divided by the amount of time taken to make a decision from that choice set, is the guiding factor for our feedback control model. We propose the use of optimal control design to model this dynamic change, using Ordinary Differential Equations (ODEs), and provide a framework for devising a method of automatically changing this number based on continuing measurements of user response on the website. This new model for web response is based on tests that have been developed to ascertain the model (Krishen and Nakamoto, 2009). Our ongoing work aims at devising the optimal control laws to design feedback controllers that work online, i.e., estimate the parameters and calculate the optimal number of choices simultaneously. Ultimately, we would like to build optimal feedback controllers for the web response system using online adaptation to customer specific parameters. However, the design of that system is cumbersome. We take a first step in this paper by solving optimal control problems that use the offline web response model. In other words, here, we assume that the consumer specific parameters are already estimated and all we need to obtain is the optimal number of choices.

2 Background

Initially, web site designs were mainly passive, with hyperlinks. When businesses realised the potential for e-commerce, websites started to provide interactive features to accommodate business transactions, such as sales to consumers. As uses and gratifications theory shows, this gave rise to the need to make websites responsive (and therefore more attractive) to users (Lin, 1996; Ruggiero 2000). Recently, some companies like Amazon.com have begun to design their websites dynamically, which means the information shown on the website responds to user's click patterns, more specifically, it keeps track of previous searches and offers suggestions. In order to clarify the definitions around e-commerce customisation, we will give short definitions of the various types of user-centered hypermedias and explain which one we are primarily concerned with in this research. The literature regarding reactive websites and models defines three key terms which we will now briefly describe (De Bra, 2002).

- adaptable hypermedia is tailored according to the information users provide through a profile, questionnaire, or survey, in a non-dynamic fashion
- adaptive hypermedia is a class of systems which changes according to a user's behaviour, using a predefined presentation
- Dynamic hypermedia, instead of using only a predefined set of presentations, can actually generate a new presentation from atomic items.

Our research is focused on using optimal control theory in order to generate dynamic changes to the number of choices per choice set on a website. It would thus be classified as a model for dynamic hypermedia. Notice that in this paper we start at a stage where we assume the system dynamics have been estimated and we focus on the optimal control design for the system.

2.1 Web feedback control formulations

We can identify two types of web design issues that can be formulated in terms of feedback control. We classify them as structural problems and quantitative control problems. These are defined and explained below:

- In structural problems the idea is to make the structure of the web design adapt to the user based influence. As a concrete example, we can express a hyperlink structure of a web design in terms of a directed graph. Each page is denoted by a node of the graph, and the hyperlink connections are represented by directed links on the graph. To represent a richer structure that allows not only hyperlinks but also the possibility of dynamically created pages and other features, we can represent the structure as a Finite State Machine (FSM). The work done by Perkowitz and Etzioni (1999, 2000) shows how the structure of the sites can be made automatically adaptive based on user access patterns. Hebbian learning has been used to adaptively structure websites by strengthening links that are used more frequently. This is based on mimicking neural based modeling for web adaptation (Bollen and Heylighen, 1996 and 1998). In their method, if a user moves from A to B, that link is strengthened, and due to a symmetry argument, the reverse link is also strengthened albeit by a smaller amount. New links can be formed due to the transitivity argument. We are not following this line of research in this paper.
- In quantitative control problems we attempt to control some quantitative parameter of the website. As an example, we might want to control the frequency of advertising pop-up windows on a page on various visits by a user of the web site. This is exactly our approach in this paper.

Using the above-mentioned discussion we can potentially obtain a customised model with parameters estimated per individual (consumer, browser, etc.) based on feedback control design and estimation techniques. This kind of approach has been developed by Krishen and Nakamoto (2009). After obtaining such a model, it would be legitimate to ascertain its practical application. This is the exact purpose of the current paper. We start at a stage where we assume the system dynamics have been obtained and we focus on optimal control design for the system. For more details on optimal control theory, see Kirk (1970), Sage and White (1977), Kamien and Schwartz (1981), and Sethi and Thompson (2000). In general, we propose a certain set of objectives that we want to achieve, which are formulated as cost minimisation or performance maximisation problems. This class of problems will be formulated as an optimal control problem of maximising an objective function subject to certain system dynamics and a set of inequality constraints that our system should satisfy. This approach is considered in the next section. We start with posing and solving a very simple model, which is afterward generalised. Then, we present an example of the general model and solve it numerically.

3 The model

Suppose that using the feedback control and estimation method, we have identified the dynamics of our system. These dynamics should match the actual behaviour of consumers. There are three important variables which we will discuss in this section. These variables are the real information rate, the nominal information rate, and the number of choices. We have devised an experimental paradigm to measure the nominal behaviour of consumers. The dynamics introduced here are completely compatible with the experimental results obtained in Krishen and Nakamoto (2009) and thus it is reasonable to assume that those dynamics are already estimated and given. Therefore, we move past the system identification stage.

3.1 The simplest model

In this paper, the main focus is on designing a framework for finding the optimal number of choices and to achieve that goal we start with a scalar ODE of the following form that constitutes our simplest model:

$$\frac{dx}{dt} = \dot{x}(t) = -\left(x - \frac{k_1 u(t)}{u^2(t) + (k_2)^2}\right), \quad \forall u > 0 \tag{1}$$

where x is the real information rate which is defined as the rate of information processed by a consumer on a web-based shopping platform, $k_1, k_2 > 0$ are constants (estimated for this specific consumer), and $u > 0$ is the number of items we decide to show to the consumer at any point of time (number of options). k_1, k_2 are the consumer specific parameters that are estimated using the suggested method in Krishen and Nakamoto (2009). In this paper, we consider k_1, k_2 as given parameters and our goal is focused on finding the optimal u. The idea behind this model is parsimonious: if the total number of choices is small, the time required to process is low, but the information processed at any step is very little too, so the overall real information rate will be small. Otherwise, if the total number of choices is too large, the time required to process is extremely high since the consumer gets overwhelmed, so the overall real information rate will also be small. Therefore, there exists an optimal number of choices (somewhere between the above extremes) that maximises the real rate of information processed by a consumer. It is important to distinguish between nominal information rate, $\dot{u}(t) = du/dt$ and the real information rate which is processed by a consumer, $x(t)$.

The steady state real information rate, x_{ss}, is calculated by setting the derivative of real information rate to zero:

$$\dot{x}(t) = 0 \Rightarrow x_{ss} = \frac{k_1 u}{u^2 + (k_2)^2}. \tag{2}$$

To get some idea of how this steady state changes as parameters k_1, k_2, and the control action u change, we have plotted x_{ss} vs. u for three sets of parameters.

In Figure 1, solid curve represents x_{ss} vs. u for $k_1 = 1, k_2 = 1$, '^' represents x_{ss} vs. u for $k_1 = 3, k_2 = 1$ and '+' represents x_{ss} vs. u for $k_1 = 1, k_2 = 3$. Figure 1 shows an interesting pattern: the peak of steady state x occurs when $u = k_2$.

Let us proceed with a closer look at the same graph in Figure 2.

At this stage it is just a conjecture. In what follows, we formally prove that this has to be the case, i.e., it is necessary and sufficient.

Figure 1 Steady state for real information rate as a function of number of options and model parameters

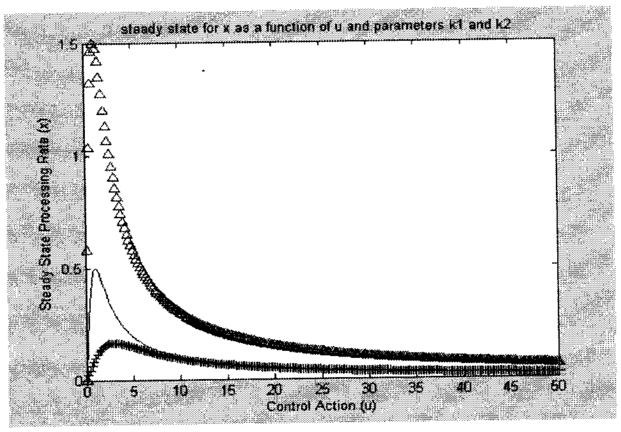
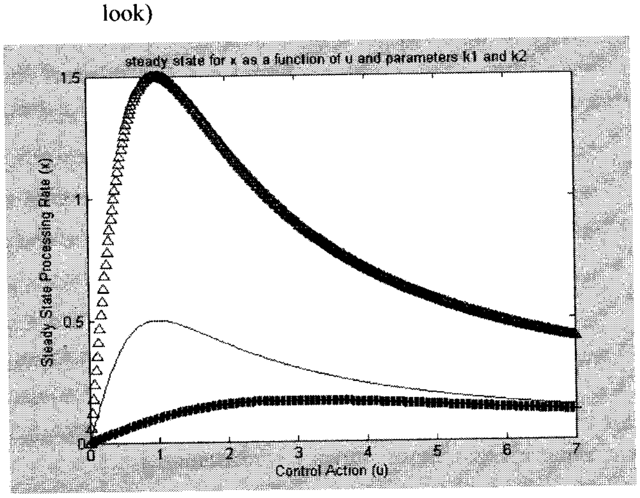


Figure 2 Steady state for real information rate as a function of number of options and model parameters (a closer look)



Our problem is that given dynamics (1), we want to maximise the rate of information processed by the consumer. Thus, we formulate the problem as follows.

$$\text{Max } J = \int_0^{t_f} x(t)dt \quad \text{subject to:}$$

$$\frac{dx}{dt} = \dot{x}(t) = -\left(x - \frac{k_1 u}{u^2 + (k_2)^2}\right), \quad \forall u > 0 \text{ and } x(0) = x_0. \tag{3}$$

Clearly, using the familiar notation used in the literature and assuming that $t_0 = 0$ and t_f is fixed,

$$\frac{dx}{dt} = \dot{x}(t) = -\left(x - \frac{k_1 u}{u^2 + (k_2)^2}\right) = a(t) \quad \text{and}$$
$$g(x,u,t) = g(x) = x(t). \tag{4}$$

The Hamiltonian is:

$$\mathcal{H}(x,p,u) = g(x) + p(t)a(t)$$
$$= x(t) - p(t)\left(x(t) - \frac{k_1 u(t)}{u^2(t) + (k_2)^2}\right). \tag{5}$$

The variational approach leads to the following equations that are the necessary conditions for an extremal:

$$\dot{p}(t) = -\frac{\partial \mathcal{H}(x,p,u)}{\partial x} = -1 + p(t) \tag{6}$$

$$\dot{x}(t) = \frac{\partial \mathcal{H}(x,p,u)}{\partial p} = -\left(x(t) - \frac{k_1 u(t)}{u^2(t) + (k_2)^2}\right) \tag{7}$$

$$\frac{d\mathcal{H}(x,p,u)}{\partial u} = k_1 \frac{(k_2)^2 - u^2}{\left[u^2 + (k_2)^2\right]^2} = 0 \Rightarrow \boxed{u*(t) = k_2}, \forall t. \tag{8}$$

If

$$u*(t) = k_2 \quad \text{then} \quad x_{ss} = \frac{k_1 u}{u^2 + (k_2)^2} = \frac{k_1}{2k_2}. \tag{9}$$

One can verify the above result by looking at Figures 1 and 2. For example for $k_1 = 1, k_2 = 1$ (the solid line) $x_{ss} = k_1 / 2k_2 = 0.5$. Since $x_{ss} = k_1 u/(u^2 + (k_2)^2)$, one could find the control action that maximises the steady state information rate as follows:

$$\frac{d}{du} x_{ss} = \frac{d}{du}\frac{k_1 u}{u^2 + (k_2)^2} = k_1 \frac{(k_2)^2 - u^2}{\left[u^2 + (k_2)^2\right]^2} = 0$$
$$\Rightarrow u(t) = k_2. \tag{10}$$

By checking the second order condition we can see if this is a maximum (<0) or minimum (>0):

$$\frac{d^2}{du^2} x_{ss} = \frac{d}{du} k_1 \frac{(k_2)^2 - u^2}{\left[u^2 + (k_2)^2\right]^2} = \frac{-4k_1 (k_2)^2 u}{\left[u^2 + (k_2)^2\right]^3} < 0. \tag{11}$$

So, $u(t) = k_2$ maximises the steady state real information rate. Also, if $a(t)$ and $g(t)$ are both concave, the necessary conditions for a maximum are sufficient, too (Kamien and Schwartz, 1981, pp.122, 123). Since $g(t)$ is linear, it is both concave and convex and so we only need to check if $a(t)$ is concave. Indeed, $a(t)$ is linear in x and concave in u, and this means $u(t) = k_2$ is the maximiser of the objective function (see the Appendix and also note that second derivative of the second term of system dynamics (1) with respect to u is negative).

This result matches the result we had based on the effect of u on the steady state x. Since there is a closed form for the optimal controller, which incidentally is an open loop solution (because it is not a function of state), there is no need to use any numerical technique. Also, by assuming that $k_2 > 0$, we know that $u^*(t) = k_2 > 0$. Hence, there is no need to apply any non-negativity constraint on $u(t)$. Also, for a given $u(t)$ the ODE above is a linear one. We know the general form of the solution (exponential) and therefore, for admissible initial state, we get the admissible state (positive real information rate). We have simulated the model for $k_1 = 5, k_2 = 5, x_0 = 0.1$, with the optimal control $u^*(t) = k_2 = 5$ (solid line in Figure 3), together with two other values for u, namely $u_1(t) = 7$ (with '*' in Figure 3), and $u_1(t) = 3$ (with '.' in Figure 3). It clearly shows the optimal solution is given by $u^*(t) = k_2 = 5$.

Figure 3 The system response to constant *u* at different levels of *u*

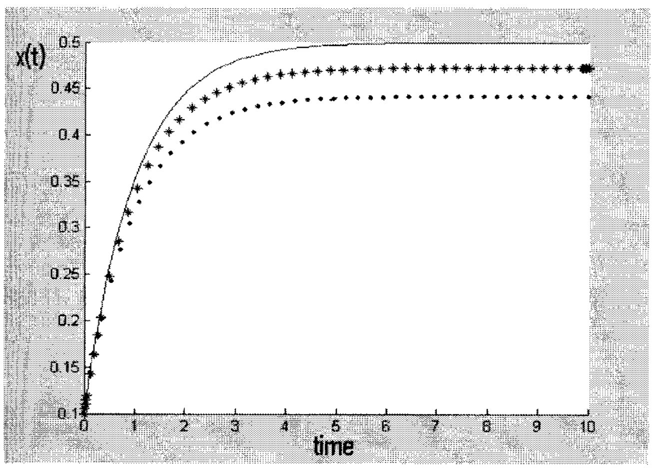
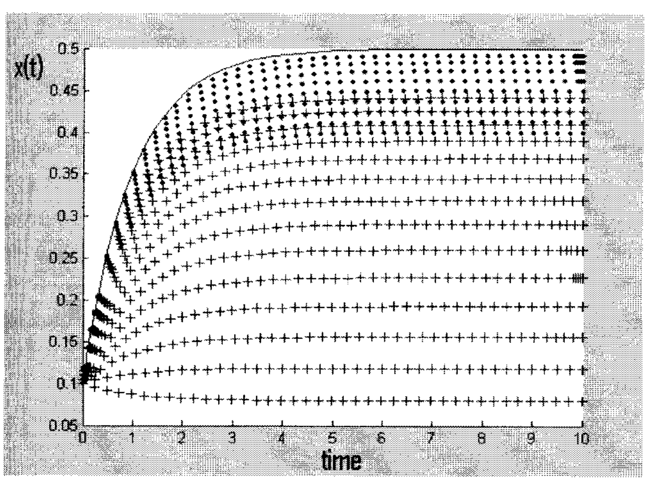


Figure 4 shows the same fact but uses a wide range for *u* (solid line for the optimal, '.' for $u > 5$, and '+' for $u < 5$):

Figure 4 The system response to constant u for a wide range of *u*'s



Similar graphs can be generated for different initial conditions that we have not reported here.

3.2 *The general model*

Now, we generalise our model to the case of maximising a given function of the real information rate (x_1) at the same time as minimising a function of the rate of change of the number of options $(v = \dot{u})$. In general, changing the number of options is not desirable so we are adding a second term to our objective function to incorporate this goal. In addition, we assume that the real information rate (x_1) is greater than a items per period $(x_1 \geq a)$. In addition to our primary task of the simple model, now we want to make it possible to use the numerical technique of variation of extremals to solve the two-point boundary value problem. In the current problem, the control variable (here, it means v) should be calculated as a function of states and co-states, i.e., $v(t) = r(x(t), p(t), t)$. We maximise the new performance measure, J:

$$\operatorname{Max} J = \int_0^{t_f} \left(h(x_1(t)) + f(v(t))\right) dt \tag{12}$$

subject to the system dynamics (13). Further, we assume that f is well-defined on its domain,[1] $f'(v) < 0$ and f' is invertible. Also, we assume that $h(.)$ and $f(.)$ are defined so that the necessary conditions are also sufficient conditions. The idea behind this objective function is exactly the same as the previous one with some computational considerations in mind. We want to calculate $v(t)$ as a function of $x(t)$ and $p(t)$ so that we can create the solution of this two-point boundary value problem using a numerical technique like variation of extremals. This aspect facilitates obtaining a numerical solution. The system dynamics are given below:

$$\begin{cases} \dot{x}_1(t) = -\left(x_1(t) - \dfrac{k_1 x_2(t)}{(x_2(t))^2 + (k_2)^2}\right), \\ \dot{x}_2(t) = v(t) \\ \dot{x}_3(t) = -(x_2(t) - a)^2 \mathbb{U}(-x_2(t) + a) \end{cases} \tag{13}$$

where

$$\mathbb{U}(-x_2 + a) = \begin{cases} 0 & x_2 \geq a \\ 1 & x_2 \leq a \end{cases} \tag{14}$$

is the unit Heaviside step function. By adding x_3 as a state variable, we are imposing a condition that the number of options is greater than or equal to a $(x_2 \geq a)$. See Kirk (1970, pp.237–240) for a detailed discussion about this method. Moreover, for applying a non-negativity constraint on the number of options shown to the customer, let $a = 0$. The initial conditions are

$$x_1(t_0) = x_{10}, \quad x_2(t_0) = a, \quad x_3(t_0) = 0. \tag{15}$$

In addition, we must have $x_3(t_f) = 0$. Define Hamiltonian function as:

$$\mathcal{H}(x,p,v) = h(x_1(t)) + f(v(t)) - p_1(t)\left(x_1(t) - \frac{k_1 x_2(t)}{(x_2(t))^2 + (k_2)^2}\right) + p_2(t)v(t) - p_3(t)(x_2(t)-a)^2\mathbb{U}(-x_2(t)+a). \tag{16}$$

Using this function by applying the necessary conditions for the optimal control, we will get the following Hamiltonian dynamics besides the system dynamics (13):

$$\dot{p}_1(t) = -\frac{\partial h(x_1(t))}{\partial x_1} + p_1(t) = -s(x_1(t)) + p_1(t) \tag{17}$$

$$\dot{p}_2(t) = -k_1 p_1(t)\left\{\frac{(k_2)^2 - (x_2(t))^2}{\left[(x_2(t))^2 + (k_2)^2\right]^2}\right\} + 2p_3(t)(x_2(t)-a)\mathbb{U}(-x_2(t)+a) + 2p_3(t)(x_2(t)-a)^2\delta(-x_2(t)+a) \tag{18}$$

where $\delta(.)$ is the Dirac's delta function with the fundamental property that for any small $\varepsilon > 0$, we have:

$$\int_{a-\varepsilon}^{a+\varepsilon} f(x)\delta(x-a)dx = f(a). \tag{19}$$

Note that the term with $\delta(.)$ is zero distribution, because $\delta(.)$ is always zero but at $x_2 = a$ and when $\delta(.)$ is nonzero, $(x_2(t)-a)$ is zero. For a reference, see the footnote on page 240, Kirk (1970). Hence, equation (18) can be simplified as,

$$\dot{p}_2(t) = -k_1 p_1(t)\left\{\frac{(k_2)^2 - (x_2(t))^2}{\left[(x_2(t))^2 + (k_2)^2\right]^2}\right\} + 2p_3(t)(x_2(t)-a)\mathbb{U}(-x_2(t)+a). \tag{20}$$

Finally,

$$\dot{p}_3(t) = 0. \tag{21}$$

Initial and final conditions are given in equation (15) and:

$$p_1(t_f) = p_2(t_f) = x_3(t_f) = 0. \tag{22}$$

Also, we need to have

$$\frac{d\mathcal{H}(x,p,v)}{\partial v} = -\frac{\partial f(v(t))}{\partial v} + p_2(t) = 0 \Rightarrow p_2(t) = f'(v). \tag{23}$$

Assuming that f' is invertible one can solve for v and calculate $v(t)$ as a function of $p_2(t)$. Let us name the inverse function w. Hence,

$$v(t) = w(p_2).^2 \tag{24}$$

This will help us yield the state dynamics (13) (noting that given equation (24) we have $\dot{x}_2(t) = w(p_2)$) and co-state dynamics (17), (20) and (21). Moreover, initial and final conditions are given in equations (15) and (22).

The above formulation is known as a nonlinear two-point boundary value problem that can be solved numerically. One can use the variation of extremals described in Kirk (1970) to solve this problem and compute the optimal control numerically for a given functional form $h(x_1)$ and $f(v)$. Also, in choosing these functional forms, one should check for the sufficient condition introduced in Theorem 2.1 in Sethi and Thompson (2000).

3.3 *An example for the general model*

Now, let us consider an example of the above general model. Assume that we are interested in minimising the change in control action, as well as maximising the information rate processed by the customers. In that case, we want to treat *u(t)* as a state variable. This changes the order of the original system given in equation (1) from one to two. Thus, our model changes to:

$$\begin{cases} \dfrac{dx_1(t)}{dt} = \dot{x}_1(t) = -(x_1(t) - \dfrac{k_1 u(t)}{(u^2) + (k_2)^2}), \quad \forall u > 0 \\ x_2(t) = u(t) \Rightarrow \dfrac{dx_2(t)}{dt} = \dot{x}_2(t) = \dot{u}(t) = v(t) \end{cases} \tag{25}$$

We can rewrite the system dynamics as follows where $\dot{u}(t) = v(t)$ is the new control action (in other words, we are adding an integrator to the system):

$$\begin{cases} \dot{x}_1(t) = a_1(t) = -\left(x_1(t) - \dfrac{k_1 x_2(t)}{(x_2(t))^2 + (k_2)^2}\right). \\ \dot{x}_2(t) = a_2(t) = v(t) \end{cases} \tag{26}$$

The above system is the same as our general model. Suppose that we want to maximise performance function, *J*:

$$\text{Max}\, J = \int_0^{t_f} \left(x_1(t) - v^2(t)\right)dt \tag{27}$$

so, $h(x_1) = x_1$ and $f(v) = -v^2$ in this example.

The idea here is to minimise the rate of change of *u*, (note that $x_2(t) = u(t)$ so, $v(t)$ is the nominal information rate), when maximising the real information rate processed by a customer, i.e., $x_1(t)$. That is why we have chosen a negative quadratic form $-v^2(t)$ in the objective function. The negative sign is there to make sure we are minimising since maximising the negative of a variable is equivalent to minimising that variable. The term $v^2(t)$ guarantees that we can approach the desired number of options (and therefore information rate) from below or above, recalling that $v(t)$ is the change in the number of options which can be positive, negative, or zero. Initial conditions are:

$$x(0) = x_0 = \begin{bmatrix} x_{10} \\ 1 \end{bmatrix}, \tag{28}$$

and note that $g(x,v) = x_1(t) - v^2(t)$.

Consequently, we can write:

$$\begin{aligned}\mathcal{H}(x,p,v) &= g(x,v) + p_1(t)a_1(t) + p_2(t)a_2(t)\\ &= x_1(t) - v^2(t) - p_1(t)x_1(t) + \frac{k_1 p_1(t) x_2(t)}{(x_2(t))^2 + (k_2)^2}\\ &\quad + p_2(t)v(t).\end{aligned} \tag{29}$$

So, our model is consisting of equations (26) as well as the following necessary conditions:

$$\dot{p}_1(t) = -\frac{\partial \mathcal{H}(x,p,u)}{\partial x_1} = -1 + p_1(t) \tag{30}$$

$$\dot{p}_2(t) = -\frac{\partial \mathcal{H}(x,p,u)}{\partial x_2} = -p_1(t)k_1\left\{\frac{(k_2)^2 - (x_2(t))^2}{\left[(x_2(t))^2 + (k_2)^2\right]^2}\right\} \tag{31}$$

$$0 = \frac{d\mathcal{H}(x,p,u)}{\partial v} = -2v(t) + p_2(t) \Rightarrow p_2(t) = 2v(t). \tag{32}$$

From the equation (30) we conclude that:

$$p_1(t) = e^{t-t_0}p_{10} - e^{t-t_0} + 1 \tag{33}$$

where $p_1(0) = p_{10}$. Using the final condition on $p_1(t)$,

$$p_1(t_f) = 0 \Rightarrow p_1(t_f) = e^{t_f - t_0}p_{10} - e^{t_f - t_0} + 1 = 0 \tag{34}$$

which leads to

$$e^{t_f - t_0}\left(p_{10} - 1\right) + 1 = 0 \Rightarrow \boxed{p_{10} = -e^{t_0 - t_f} + 1}. \tag{35}$$

Substitute p_{10} in $p_1(t)$

$$p_1(t) = e^{t-t_0}\left(-e^{t_0 - t_f} + 1\right) - e^{t-t_0} + 1 = 1 - e^{t-t_f}. \tag{36}$$

From equations (31) and (36) we get:

$$\dot{p}_2(t) = -k_1(1 - e^{t-t_f})\left\{\frac{(k_2)^2 - (x_2(t))^2}{\left[(x_2(t))^2 + (k_2)^2\right]^2}\right\}. \tag{37}$$

Using equation (26) and differentiating equation (32) with respect to time:

$$p_2(t) = 2v(t) = 2\dot{x}_2(t) \Rightarrow \dot{p}_2(t) = 2\ddot{x}_2(t). \tag{38}$$

Now, substituting equation (38) in equation (37) leads to the following second order ODE that can be solved numerically given the initial conditions,

$$\ddot{x}_2(t) = -\frac{k_1}{2}(1 - e^{t-t_f})\left\{\frac{(k_2)^2 - (x_2(t))^2}{\left[(x_2(t))^2 + (k_2)^2\right]^2}\right\}. \tag{39}$$

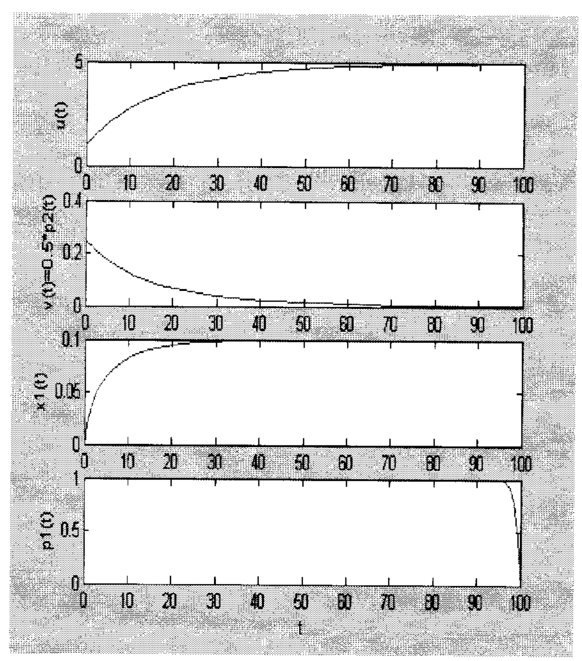
Assuming $x_2(t) = y_1(t)$, $\dot{x}_2(t) = (1/2)p_2(t) = v(t) = y_2(t)$, $x_1(t) = y_3(t)$, $p_1(t) = y_4(t)$,, and we have simulated the following system of equations in MATLAB® (given the above-mentioned initial and final conditions):

$$\begin{cases}\dot{y}_1(t) = y_2(t)\\ \dot{y}_2(t) = -\dfrac{k_1 y_4(t)}{2}\left(\dfrac{(k_2)^2 - (y_1(t))^2}{\left[(y_1(t))^2 + (k_2)^2\right]^2}\right)\\ \dot{y}_3(t) = -\left(y_3(t) - \dfrac{k_1 y_1(t)}{(y_1(t))^2 + (k_2)^2}\right)\\ \dot{y}_4(t) = -1 + y_4(t)\end{cases}. \tag{40}$$

In the above formulation, we have not imposed a non-negativity constraint. However, from the simulation results, we will see that the constraint was satisfied, without enforcing it, i.e., states remain non-negative. In order to perform our simulation, we had to solve the above two-point boundary value problem numerically. Using the function 'bvp4c' in MATLAB®, we have completed the simulation. See Shampine et al. (2003) for a tutorial on how to use 'bvp4c'. For a detailed description of the 'bvp4c' function, its arguments, and outputs consult MATLAB® reference and user guide which is also available online (see documentation for MathWorks products, R2008b). We have performed the simulation for a range of parameters including the ones that we used in our simulation for the first model (Figures 1–4).

We are not reporting the simulation results for all these simulations here. Figure 5 shows the simulation results for a new set of parameters, initial and final conditions. The most interesting observation here is that the optimal number of options in the steady state is still equal to k_2. Moreover, the steady state real information rate processed by customers is $k_1/2k_2$. These values are consistent with the results of our first model.

Figure 5 Simulation results for: $k_1 = 1, k_2 = 5, t_0 = 0, t_f = 100, u(0) = 1, x_1(0) = 0, p_1(100) = 0$ (see online version for colours)



4 Conclusion

In this paper, we have presented a description of how the steps of feedback control design and optimal control theory can be applied to the dynamic design of websites. We did so by making the number of options shown to a consumer a control variable that depends on how many items per period is processed by that individual. Internet shopping platforms can adopt this framework to adjust their website design with consumers' ability of processing information. We considered a performance measure that maximises the real information rate as a state variable and at the same time minimises the rate of change in number of options available to the consumer. We used a creative form of system dynamics for the real information rate and applied the optimal control theory to find the optimal number of options that should be shown to the consumer. We obtained the necessary conditions for the optimal control for a class of objective functions. We showed that the resulting nonlinear two-point boundary value problem can be solved numerically using the method of variation of extremals. We ran simulations for some given sets of model parameters and observed that our results are consistent with the predictions of our simplest model. For future work, we plan to integrate the estimation of consumer specific parameters and determination of optimal control. This makes the online optimal adaptation of dynamic websites possible.

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References

Bollen, J. and Heylighen, F. (1996) 'Algorithms for the self-organization of distributed, multi-user networks. Possible application to the future world wide web', in Trappl, R. (Ed): *Cybernetics and Systems '96,* Austrian Society for Cybernetic Studies, Vienna, Austria, pp.911–916.

Bollen, J. and Heylighen, F. (1998) 'A system to restructure hypertext networks into valid user models', *New Review of Hyper Media and* Multimedia, Vol. 4, pp.189–213.

Bucklin, R.E. and Sismeiro, C. (2003) 'A model of web site browsing behavior estimated on clickstream data', *Journal of Marketing Research*, Vol. 40, No. 3, p.249.

Chi, E.H., Pirolli, P. and Pitkow, J. (2000) 'The scent of a site: a system for analyzing and predicting information scent, usage, and usability of a web site', *Proceeding of ACM Conference on Human Factors in Computing Systems (CHI 2000)*, The Hague, Netherlands, 1–5 April, ACM, NY, pp.161–168.

Chi, E.H., Pirolli, P., Chen, K. and Pitkow, J. (2001) 'Using information scent to model user information needs and actions on the web', *Proceeding of ACM Conference on Human Factors in Computing Systems (CHI 2001)*, Seatle, WA, 31 March–5 April, ACM, NY, pp.490–497.

De Bra, P. (2002) 'Adaptive educational hypermedia on the web. Association for computing machinery', *Communications of the ACM*, Vol. 45, No. 5, p.60.

Heylighen, F. and Bollen, J. (2002) 'Hebbian algorithms for a digital library recommendation system', *Proceedings of 2002 International Conference on Parallel Processing Workshops,* IEEE Computer Society, Vancouver, BC, Canada, pp.439–446.

Kamien, M.I. and Schwartz, N.L. (1981) *Dynamic Optimization: The calculus of Variation and the Optimal Control in Economics and Management*, North-Holland, New York.

Kirk, D.E. (1970) *Optimal Control Theory: An Introduction*, Prentice-Hall, Inc., Englewood Cliffs.

Krishen, A.S. and Nakamoto, K. (2009) 'Improving consumer quality-efficiency by using simple adaptive feedback in a choice setting', *International Journal of Computer Applications in Technology*, Vol. 34, No. 3, pp.155–164.

Lin, C. (1996) 'Looking back: the contribution of blumler and katz's uses of mass communication to communication research', *Journal of Broadcasting and Electronic Media*, Vol. 40, No. 4, pp.574–581.

Park, Y.H. and Fader, P.S. (2004) 'Modeling browsing behavior at multiple websites', *Marketing Science*, Vol. 23, No. 3, p.280.

Perkowitz, M. and Etzioni, O. (1999) 'Towards adaptive web sites: conceptual framework and case study', *Computer Networks*, Vol. 31, Nos. 11–16, p.1245.

Perkowitz, M. and Etzioni, O. (2000) 'Adaptive web sites. Association for computing machinery', *Communications of the ACM*, Vol. 43, No. 8, p.152.

Ruggiero, T. (2000) 'Uses and gratifications theory in the 21st century', *Mass Communication and Society*, Vol. 3, No. 1, pp.3–37.

Sage, A.P. and White, C.C. (1977) *Optimum Systems Control*, Prentice-Hall, Englewood Cliffs.

Sethi, S.P. and Thompson, G.L. (2000) *Optimal Control Theory: Applications to Management Science*, Kluwer Academic Publishers, Boston.

Shampine, L.F., Reichelt, M.W. and Kierzenka, J. (2003) *Solving Boundary Value Problems for Ordinary Differential Equations in MATLAB® with bvp4c*, Available online at: http://www.mathworks.com/bvp_tutorial

Documentation for MathWorks Products, R2008b, *available online at*: http://www.mathworks.com/access/helpdesk/help/helpdesk.html

Notes

[1]This means $f(0) \neq \infty$, and so $u = \text{constant}$ will not lead to $J = \infty$.

, J. (2001) 'Using
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rmedia on the web.
Communications of
Iebbian algorithms
em', Proceedings of
arallel Processing
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amic Optimization:
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: An Introduction,
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3, pp.155–164.
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[2]To pursue the numerical technique (variation of extremal) we shall have $v(t) = r(x(t), p(t), t)$. Writing $d\mathcal{H} = (x, p, v) / \partial v = 0$ and the use of Implicit Function Theorem becomes the primary method to achieve this goal. However, in our formulation having an invertible f' is sufficient.

Appendix

Lemma 1: *$h(x,u) = f(x) + g(u)$ is concave if $f(x)$ and $g(u)$ are concave.*

Proof: Since $f(x)$ and $g(u)$ are both concave, for any $0 \le t \le 1$:

$$f(tx_1 + (1-t)x_2) \ge tf(x_1) + (1-t)f(x_2)$$

$$g(tu_1 + (1-t)u_2) \ge tg(u_1) + (1-t)g(u_2)$$

add the last two inequalities to get:

$$f(tx_1 + (1-t)x_2) + g(tu_1 + (1-t)u_2) \ge tf(x_1) + (1-t)f(x_2) + tg(u_1) + (1-t)g(u_2).$$

This leads to:

$$[f(tx_1 + (1-t)x_2) + g(tu_1 + (1-t)u_2)] \ge [tf(x_1) + tg(u_1)] + [(1-t)f(x_2) + (1-t)g(u_2)].$$

Hence,

$$h([tx_1 + (1-t)x_2],[tu_1 + (1-t)u_2]) \ge th(x_1, u_1) + (1-t)h(x_2, u_2).$$

This means that $h(x,u)$ is concave. Q.E.D.