Sta 731 - Homework 8

Due Tuesday, 12/4

1. Let $\{E_n\}$ be events. Verify

$$\sum_{k=1}^{n} 1_{E_k} = 1_{\bigcup_{k=1}^{n} E_k} \sum_{k=1}^{n} 1_{E_k}$$

and then, using the Schwartz inequality, prove that

$$P(\bigcup_{k=1}^{n} E_k) \geq \frac{\left(E(\sum_{k=1}^{n} 1_{E_k})\right)^2}{E(\sum_{k=1}^{n} 1_{E_k})^2}.$$

2. Suppose $\{X_n, n \geq 1\}$ is a sequence of random variables satisfying

$$P(X_n = n) = \frac{1}{n},$$

$$P(X_n = 0) = 1 - \frac{1}{n}.$$

(a) Does $\{X_n\}$ converge in probability? If so, to what? Why?
(b) Does $\{X_n\}$ converge in distribution? If so, to what? Why?
(c) Suppose in addition that $\{X_n\}$ is an independent sequence. Does $\{X_n\}$ converge almost surely? What is $\limsup_{n\to\infty} X_n$ and $\liminf_{n\to\infty} X_n$ almost surely? Explain your answer.

3. Let X and Y be independent Bernoulli random variables on a probability space $(\Omega, \mathcal{B}, P)$ with $X \stackrel{d}{=} Y$ and

$$P(X = 0) = \frac{1}{2} = P(X = 1).$$

Let $X_n = Y$ for $n \geq 1$. Show that

$$X_n \Longrightarrow X$$

but that X_n does NOT converge in probability to X.

4. If $T_n > 0$ satisfies

$$\sqrt{n}(T_n - \theta) \xrightarrow{d} N(0, \tau^2),$$

find the limiting distribution of

$$\text{(a)} \quad \sqrt{n}(\sqrt{T_n} - \sqrt{\theta}) \qquad \text{and} \qquad \text{(b)} \quad \sqrt{n}(\log T_n - \log \theta) \quad \text{for} \quad \theta > 0.$$

5. Suppose Y_s is Poisson distributed with parameters s so that

$$P(Y_s = k) = e^{-s}\frac{s^k}{k!}$$

Compute the chf of Y_s. Prove

$$\frac{Y_s - s}{\sqrt{s}} \Longrightarrow N(0, 1)$$